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Social distance, heterogeneity and social interactions

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ABSTRACT

A crucial ingredient in social interaction models is the structure of peer groups, which link individuals with similar characteristics. We propose and study a dynamic binary choice model with social interactions in which heterogeneity of peer group effects is modeled introducing diversity in individual characteristics and linking pairwise influences to a *social distance* between individuals. Our framework allows for mimetic as well as anti-mimetic interactions and a heterogeneous structure of peer groups across individuals. Dynamic equilibria are studied in the limit when the number of agents is large. We show that the model exhibits multiple equilibria resulting from conflicts between various group pressures the individuals are subjected to. We study in particular the correlation in the population at equilibrium between the characteristics of the agents and their decisions: this quantity has an interesting empirical interpretation and solves a simple analytical equation when the number of agents is large. Finally we discuss the empirical content of the model and present a consistent estimator for the parameter describing which is consistent for any typical population regardless of the structure of individual characteristics.

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While the economic approach to human behavior (Becker, 1974) has traditionally emphasized the role of individual preferences in explaining social phenomena, the sociological approach (Bourdieu, 2001; Coleman, 1990) essentially views individuals as members of groups and emphasizes the interaction among individuals and between the individual and the group as the determining factor in explaining aggregate phenomena. *Social interaction models* (Durlauf and Peyton Young, 2000; Föllmer, 1974; Schelling, 1971, 1973; Glaeser et al., 1996; Glaeser and Scheinkman, 2002; Horst and Scheinkman, 2006; Manski, 2000; Topa, 2001) have attempted to breach the gap between these two approaches by modelling individual decision making when preferences are influenced by one's social environment. Social interaction models take into account the fact that the preferences of individuals over actions can depend on the actions of other individuals in their social group, by considering that the utility U_i of an action ϕ_i for an agent i can depend on the actions $\{\phi_j, j \neq i\}$ of other agents:

$$U_i(\phi_i; \{\phi_j, j \neq i\}) \tag{1}$$

With respect to more common settings in game theory, here the emphasis is on the interplay between the heterogeneity of individual preferences and social interactions in large populations, where behavior is not strategic (see also Morris and Shin, 2005).

These studies explain the statistical behavior of aggregate outcomes in many socio-economic phenomena – in particular the "excess variance" of aggregate variables and social multiplier effects – in terms of a competition between two factors: the degree of *heterogeneity* of individual preferences and the strength of *peer group influence*. While it is possible to give

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some general results (Blume, 1993; Blume and Durlauf, 2003; Glaeser and Scheinkman, 2002) on existence and uniqueness of equilibria for specifications of the form (1), in order to obtain models with empirical content parametric specifications are desirable. A flexible class of models, introduced by Föllmer (1974) and used in recent studies (Durlauf and Peyton Young, 2000; Brock and Durlauf, 2000; Glaeser et al., 1996; Glaeser and Scheinkman, 2002; Topa, 2001), is obtained by assuming that utility functions can be decomposed into a sum of an individual (private) utility and social component, expressed as a sum of pairwise influences of other agents:

$$U_i(\phi_i; \{\phi_j, j \neq i\}) = U_i^0(\phi_i) + \sum_{j \neq i} J_{ij}\phi_i\phi_j \tag{2}$$

Here $U_i^0(\phi_i)$ represents the individual preference of agent i in absence of any social interaction. Heterogeneity of individual preferences can be introduced by modelling $U_i^0(\phi_i)$ as randomly distributed across the population, as in classical discrete choice models (Anderson et al., 1992; Manski and McFadden, 1981).

A crucial ingredient here is the specification of J_{ij} – the gain in utility of i from conforming to j– which encodes the interaction structure between individuals and interesting classes of models are obtained by parameterizing the interaction terms J_{ij} in a parsimonious way. The choice $J_{ij} = J/(N-1) > 0$ leads to global interaction models (Brock and Durlauf, 2000; Orléan, 1995) where each agent is affected by the average action of others. This choice assumes a high degree of homogeneity in the structure of *interactions* between agents which implies that the impact of an agent's action on all other agents is the same. This assumption, although mathematically convenient, implies that any "peer pressure" has to be present at the level of the whole population, which is rather unrealistic and goes against the notion of peer group. Empirical studies on social behavior in contexts ranging from neighborhood effects on youth behavior (Case and Katz, 1990; Evans et al., 1992; Jones, 1994) to spatial patterns in unemployment (Conley and Topa, 2002) show that while individuals belonging to different age classes, income levels, professional categories and cultural backgrounds may exhibit very different behaviors, there is less heterogeneity inside a given category. This fact cannot be accounted for in a homogeneous model where the notion of peer group has no theoretical counterpart.

By contrast, in local interaction models (Bisin et al., 2005; Glaeser et al., 1996; Topa, 2001) social influences are mediated by the peer group of an individual (Evans et al., 1992; Glaeser et al., 1996; Jones, 1994; Simmel, 1904). Each agent interacts with a group of "neighbors" on a graph whose links represent social relations. Here the main issue is choosing a meaningful notion of "social distance" to specify proximity. The local interaction models proposed in the literature (Brock and Durlauf, 2000; Föllmer, 1974; Glaeser et al., 1996; Topa, 2001) assume that this neighborhood structure is represented by a regular lattice $\mathbb{Z}^d$. This choice is often motivated by the availability of mathematical results in the framework of lattice models; it should be noted that these results use in an important way the notion of translation invariance in lattice models, a notion which to our knowledge has no clear economic interpretation. More importantly, such assumptions about the composition of peer groups impose a certain homogeneity on the structure of peer groups across individuals. For example, the size of the peer group of an individual is assumed to be known ($2d$ in this case).

In empirical studies peer groups are observed to vary in size and structure from individual to individual. For example Borjas (1992) assumes that persons interact with members of their own ethnic group; in a study on spatial patterns in unemployment (Conley and Topa, 2002) show that ethnic as well as other similarities in characteristics matter. Studies on the structure of social groups show a tendency of individuals to interact with other individuals with similar characteristics (Marsden, 1982, 1990), leading to the notion that similarity in characteristics should be a basis for social proximity (Akerlof, 1997). If such criteria are used as a metric for social proximity, the resulting network of social interactions will be a non-regular graph for which lattice models are a poor representation; moreover the properties of the resulting network will be dependent on the structure of the population under study and thus may vary from one population to another, resulting in a *random* graph of which a given population represents a sample.

The composition of "peer groups" is a crucial ingredient of social interaction models. One is then led to ask how local interactions may affect equilibrium outcomes when the size and structure of the peer groups is heterogeneous across individuals, as is the case empirically.

A related question is that of the empirical content of these models (Manski, 2000): is it possible to infer certain aggregate socio-economic outcomes given statistical information on the structure of social groups, as in population surveys? Equilibrium outcomes of the model can be meaningfully compared to empirical data only if (certain) aggregate variables possess a behavior which becomes sample-independent for large samples. In other words, the necessary condition for inference in these models is a law of large numbers for certain aggregate variables. Given the presence of social interactions, individual decisions will not be independent so obtaining a "law of large numbers" is not obvious but possible under some conditions if the peer group structure is known (Föllmer, 1974). When the structure of peer groups and interactions within them vary randomly across individuals, one needs to distinguish aggregate outcomes which depend on the detailed knowledge of this microstructure from those which only require to know the statistical features of social groups. Obviously, only the latter can be compared with survey data on large populations.

Finally, we note that a constant feature of the previously studied models is that social interactions introduce strategic complementarity: social pressure is assumed to favor imitation of others ($J_{ij} > 0$).However social interactions can tilt preferences in the opposite direction: (Simmel, 1904) highlights the roles of both imitation and the need to make distinctions in fashions and fads; (Glaeser et al., 1996) underline the presence of negative interactions among criminals due to competition for 'resources'; (Conley and Topa, 2002) note the presence of negative correlations in unemployment rates of individuals when ethnic differences are large. It is therefore interesting to allow for negative as well as positive social influences in social interaction models.

Our intent here is to propose a modeling framework in which the above issues have meaningful theoretical counterparts. We propose and study a dynamic binary choice model with social interactions in which heterogeneity is introduced at two different levels: at the level of agents preferences by introducing an agent-specific random component in the utility function, and at the level of the interaction structure by taking into account affinities between agents with similar characteristics, allowing for positive as well as negative interactions as well as a heterogeneous structure of peer groups across individuals.

The introduction of a dynamic framework allows us to distinguish between the two types of randomness: behavioral heterogeneity, stemming from randomness in individual preferences, modeled as IID across time and randomness in population characteristics and peer group structure, which does not vary in time but only across populations.[1] We study dynamic equilibria of our model and describe them through the level of conformity of individuals decisions to their characteristics.

Our dynamic setting is similar to stochastic choice revision schemes studied in evolutionary game theory (see e.g. Baron et al., 2004 or Young, 1998, Chapter 6 for a review) in which one is interested in stochastically stable states (Foster and Young, 1990) obtained by studying the behavior of equilibria when randomness is driven to zero. However in our case randomness constitutes a behavioral component of interest, corresponding to bounded rationality in agent behavior, and not just an equilibrium selection scheme.

This article is organized as follows. Section 1 provides a mathematical description of our model in terms of agents characteristics, communication structure, preferences, decision rules and dynamics. In Section 2 we define static and dynamic equilibria and study their properties. In Section 2.3 we show existence and uniqueness of a dynamic equilibrium *given* the characteristics of the population and describe these equilibria through the correlations between agents characteristics and their choices. While the dynamic equilibria clearly depend on the characteristics of the individuals in the population, in Section 3 we show that if the population is large then the correlations between individuals actions and their characteristic can only have a finite number of configurations, thus obtaining a result robust to the randomness of population characteristics. Exploiting this result we explore the empirical content of our model and propose in Section 3.2 a method for measuring the heterogeneity of individual preferences, based on aggregate observations on choices of individuals. Section 4 discusses how these results are modified by the presence of a bias in individual preferences, which can result among other things from incentives provided by policy maker. Finally, Section 5 concludes by discussing the obstacles to reaching equilibrium in large heterogeneous populations and its implications for the empirical content of social interaction models.

1. Model framework

We now introduce a model describing the influence of social interactions on the decision making behavior of a large number of heterogeneous agents, facing a binary choice problem, emphasizing the economic interpretation of the concepts introduced.

A description of agents by their characteristics is introduced in Section 1.1. Section 1.2 defines the notion of social affinity between agents. The communication structure among the agents is described in Section 1.3. Section 1.4 describes the agents preferences and their interactions. Section 1.5 discusses the probabilistic description of the agents decision rules. Section 1.6 describes some quantities of interest that we focus on in the rest of the article.

1.1. Heterogeneity of characteristics and social groups

Consider a set of agents, labelled by $i = 1 \dots N$, where each agent is described in terms of M possible attributes such as age, gender, income level, ethno-cultural background, etc. Such descriptions are precisely of the type given in population surveys in which individual profiles are specified through multiple-choice questions asking to identify the age category, income category, ethnic group, etc. Characteristics will be denoted by the Greek letter α. For example $\alpha = 1$ can correspond to the age category, $\alpha = 2$ to gender, $\alpha = 3$ to the income category, $\alpha = 4$ to ethnic group, etc. For simplification we shall assume that these attributes are "binary"—for each characteristic α, there are two exclusive possibilities e.g. young/old, male/female, high income/ low income, ... The attributes of agent i can then be described by a sequence $(\xi_i^\alpha)_{\alpha=1\dots M}$ of binary variables, indicating whether agent i possesses the characteristic ($\xi_i^\alpha = \xi_+^\alpha > 0$) or not ($\xi_i^\alpha = \xi_-^\alpha < 0$).

A *social group* can then be defined as a set of individuals sharing the same characteristics. Each social group can therefore be described as an element of $\prod_{\alpha=1}^M \{\xi_-^\alpha, \xi_+^\alpha\}$ while the statistical description of a heterogeneous population is given by a convex combination of these elements i.e. a vector in $\prod_{\alpha=1}^M [\xi_-^\alpha, \xi_+^\alpha]$, the barycentric coordinates describing the proportions of each social group in the population.

In large heterogeneous populations, only a statistical description of population characteristics is typically available. For example, we might know that a population is 40% young, 55% female, 70% white, etc.

[1] While this distinction is also made by Horst and Scheinkman (2006), their setting is a static one in which the distinction does not entail any consequence.

Heterogeneity of characteristics can then be modelled by assuming that for each characteristic $\alpha = 1 \ldots M$ the binary variables $(\xi_i^\alpha)_{i=1\ldots N}$ form a family of independent identically distributed (IID) random variables such that:

$$\mathbb{P}(\xi_i^\alpha = \xi_+^\alpha) = 1 - \mathbb{P}(\xi_i^\alpha = \xi_-^\alpha) = \pi^\alpha \qquad N \geq i \geq 1, \quad M \geq \alpha \geq 1. \tag{3}$$

where π^α is the fraction of the population with characteristic $\xi^\alpha > 0$. Thus, a population of size N is represented by a sample $(\xi_i)_{i=1\ldots N}$ of IID draws from the distribution above. For each individual, $\xi_i = (\xi_i^\alpha, \alpha = 1 \ldots M)$ describes the attributes of i.

Here ξ_i^α are simply indicator variables and their values $\xi_+^\alpha, \xi_-^\alpha$ may be normalized such as to center them:

$$\xi_+^\alpha = 2(1-\pi^\alpha)\xi_-^\alpha = -2\pi^\alpha \tag{4}$$

In the particular case where $\pi^\alpha = 1/2$ then ξ_i^α takes the values ± 1.

1.2. Social affinity

As argued above, one of the main issues in a model of social interactions is to define a notion of 'social distance' between agents. Although in some applications of local interaction models (Glaeser et al., 1996; Topa, 2001) this notion has been assimilated to a spatial distance, as noted in the introduction the notion of social distance is not limited to geographical distance (Akerlof, 1997). For example (Conley and Topa, 2002) show that "ethnic distance clearly seems to be dominant in terms of explaining" spatial patterns in unemployment. More generally, sociological studies (Marsden, 1982, 1990) indicate the tendency of individuals to interact more with other individuals with similar characteristics. These observations motivate the following definition: given a description of agents by the vector of their characteristics $\xi_i = (\xi_i^\alpha)_{\alpha=1\ldots M}$, we define the social affinity of agents i and j is given by the scalar product of ξ_i and ξ_j:

$$a(i,j) = \sum_{\alpha=1}^{M} \xi_i^\alpha \xi_j^\alpha \tag{5}$$

Two agents with identical profiles have maximal affinity while two agents with opposite profiles have *negative* affinity. In the case where all $\pi^\alpha = 1/2$, $a(i,j)$ simply reduces to the number of common attributes i and j have in common, opposite attributes being counted negatively. If agents i and j have many common attributes, then $a(i,j)$ is large and positive. In a homogeneous population, all pairs of agents have the same affinity. In a heterogeneous population with characteristics randomly distributed across the population, the affinity between two individuals will also be randomly distributed across pairs of agents.

Recall that in order to center the variable ξ_i^α we normalized its values to $\xi_+^\alpha = 2\pi^\alpha$, $\xi_-^\alpha = 2(1-\pi^\alpha)$. Now assume $\pi^\alpha \geq 1/2$ (the majority of the individuals have characteristic $\xi_i^\alpha > 0$) and take i, j belonging to the majority ($\xi_i^\alpha = \xi_j^\alpha = 2(1-\pi^\alpha)$) and i', j' belonging to the minority $\xi_{i'}^\alpha = \xi_{j'}^\alpha = -2\pi^\alpha$. Then

$$\xi_i^\alpha \xi_j^\alpha < \xi_{i'}^\alpha \xi_{j'}^\alpha.$$

which means that belonging to the same minority has a larger impact on social affinity than belonging to the same majority: peer pressure is stronger on members of smaller social groups.

1.3. Random communication structure

In standard game theoretical models, all agents communicate with each other: the communication structure is described by a complete graph linking all pairs $\{i,j\}$. In a large population it is more realistic to assume that each agent can communicate with some, but not all, other agents. This can be modeled by a (Erdös-Renyi) random graph: given a pair of agents $\{i,j\}$, there is a probability p that they communicate, independently from all other pairs. The communication structure may then be described as in Kirman (1983) via a random graph with N vertices and IID edges. Define the set of links (edges) by

$$L_N = \{\{i,j\} | 1 \leq i \leq N, 1 \leq j \leq N, i \neq j\} \tag{6}$$

Defining an indicator variable $\varepsilon_{ij} \in \{0,1\}$ for each pair of agents, a random communication structure can be represented by a random variable $\varepsilon = (\varepsilon_{ij}, i \neq j)$ taking values in $\{0,1\}^{L_N}$ with IID components such that:

$$\mathbb{P}(\varepsilon_{ij} = 1) = 1 - \mathbb{P}(\varepsilon_{ij} = 0) = p \tag{7}$$

For $p < 1$ a completely connected graph is an atypical configuration in a large population: the probability that all pairs of agents communicate is then $p^{N(N-1)/2}$, which tends to zero when N is large.

1.4. Preferences

Each agent is facing a choice between between two feasible alternatives: the action of each agent i can be represented by a variable $\phi_i \in S = \{-1, +1\}$. For example, in the classical setting of the discrete choice theory of product differentiation (Anderson et al., 1992) agents may be faced with a choice beween product brands. Another example is that of a speculative market for a financial asset (Cont and Bouchaud, 2000), where each trader is faced with the decision whether to buy or sell and eventually the size of the order. Alternatively, one may consider the choice of adopting or not adopting certain social practices: the decision to smoke (Jones, 1994), to follow a social trend (Evans et al., 1992; Simmel, 1904; Schelling, 1973), to engage in a criminal act (Glaeser et al., 1996). This formalism has been also applied to cases where ϕ_i is not the result of a decision but describes the state of the agent i as a result of interaction with his environment e.g. unemployment (Topa, 2001).

Agent i chooses ϕ_i such as to maximize a utility function U_i, which is the sum of two components: an individual (private) utility $U_i^0(\phi_i)$ which only depends on the choice ϕ_i of agent i and a *social* component V_i which also depends on the choices of the other agents $\phi_j, j \neq i$:

$$U_i(\phi) := U_i^0(\phi_i) + V_i(\phi_i, \{\phi_j, j \neq i\}) \tag{8}$$

Since $\phi_i \in \{1, -1\}$, U_i^0 can be assumed to be linear without loss of generality. As in (Brock and Durlauf, 2000) we model this component as:

$$U_i^0(\phi_i) = u_i\phi_i + h(\xi_i)\phi_i \tag{9}$$

where h is common to all agents in a social group and expresses a social bias in preferences and the idiosyncratic term u_i is specific to agent i and describes her individual taste. In Sections 2 and 3 we will set $h \equiv 0$; in Section 4 we will allow h to depend on the agents characteristics and analyze its effect on equilibrium behavior. Heterogeneity of individual tastes is modelled via a random utility approach by considering $(u_i)_{i=1\ldots N}$ as a sequence of IID variables whose distribution we denote by F:

$$F(x) = \mathbb{P}(u_i \leq x) \tag{10}$$

A common parametric form used for inference is the *logit* distribution (Anderson et al., 1992; Manski and McFadden, 1981):

$$F(x) = \frac{1}{1 + \exp(-\beta x)} \tag{11}$$

where $\beta > 0$ is a homogeneity parameter: large β indicates a population where agents share similar individual tastes while small β indicates a large heterogeneity of tastes. The logit distribution can be also derived as a tradeoff between exploration and "exploitation" where $1/\beta$ may be viewed as representing the cost of exploration (Anderson et al., 1992; Nadal and Weisbuch, 1998).

We allow the preferences of an agent i to be influenced by actions of his/her peer group ν_i, specified as being the set of agents communicating with i:

$$\nu_i = \{j | \varepsilon_{ij} = 1\} \tag{12}$$

Given the random communication structure among agents, ν_i is a *random* set and its size may vary across individuals. A flexible parametric form is obtained by assuming that V_i has an additive structure (Föllmer, 1974; Brock and Durlauf, 2000; Glaeser et al., 1996; Topa, 2001):

$$V_i(\phi_i, \{\phi_j, j \neq i\}) = \frac{1}{pN}\sum_{j \in \nu_i} J_{ij}\phi_j\phi_i \tag{13}$$

where J_{ij} is the gain in utility of agent i resulting from aligning her choice on that of agent j. Since the average size of a neighborhood ν_i is pN, the normalizing factor $1/pN$ sets the right scale[2] such that when N grows one the social component does not systematically dominate the other term in U_i. The quantity J_{ij} plays the role of an *interaction coefficient* describing the influence of the choice ϕ_j of agent j on agent i. For example if $J_{ij} > 0$, agent i tends to imitate j. $J_{ij} = 0$ means that the choice of agent j does not affect agent i. With the parametrization introduced above, the mutual influence of two agents choice is described by the interaction coefficient J_{ij}. Different specifications of J_{ij} then lead to different interaction structures.

Using the notion of social affinity defined in Section 1.2, it is natural to assume that agents with a higher social affinity will have a higher interdependence in their choices. A simple parametrization of this idea is to choose:

$$J_{ij} = \varepsilon_{ij}\, a(i,j) = \varepsilon_{ij}\sum_{\alpha=1}^{M}\xi_i^\alpha\xi_j^\alpha \tag{14}$$

The idea is that the mutual influence between two agents – parameterized here by J_{ij} – is related to their social affinity: if agents i and j share many common attributes (for example if they are both young, male and belong to the same ethnic group) the choice of j is more likely to influence that of i than if i and j share few common attributes, a fact that is corroborated by various behavioral studies (Marsden, 1982). Finally, if two agents do not communicate ($\varepsilon_{ij} = 0$) they do not influence each

[2] Here pN can be replaced by $|\nu_i|$, the (random) size of the neighborhood of i, without changing the results in the sequel.

others actions, no matter how similar or different their characteristics may be. Note that (14) implies that J_{ij} are random variables.

1.5. Probabilistic description of individual choice

Each individual i makes her choice ϕ_i by maximizing her utility function U_i as in (8). Since the utility of an action for i depends on action of other individuals $j \in \nu_i$ in her peer group, this utility maximization problem is only well defined conditionally on the variables $(\phi_j)_{j\in\nu_i}$. Given $(\phi_j)_{j\in\nu_i}$, i will choose $\phi_i = +1$ if

$$U_i^0(+1) + V_i(+1, \{\phi_j, j \in \nu_i\}) > U_i^0(-1) + V_i(-1, \{\phi_j, j \in \nu_i\}) \Leftrightarrow U_i^0(+1) - U_i^0(-1) > V_i(-1, \{\phi_j, j \in \nu_i\})$$
$$- V_i(+1, \{\phi_j, j \in \nu_i\}) \Leftrightarrow u_i > -h - \frac{1}{pN}\sum_{j\in\nu_i} J_{ij}\phi_j \tag{15}$$

The conditional probability of i choosing $\phi_i = 1$ is therefore given by:

$$\mathbb{P}(\phi_i = +1|(\phi_j)_{j\in\nu_i}, J) = 1 - \mathbb{P}\left(u_i \le -h - \frac{1}{pN}\sum_{j\in\nu_i} J_{ij}\phi_j\right) = 1 - F\left(-h - \frac{1}{pN}\sum_{j\in\nu_i} J_{ij}\phi_j\right) \tag{16}$$

Here the conditioning on $J = (J_{ij}, j \in \nu_i)$ means that the structure of interactions in the population is fixed. In the case where F is a logit distribution given by (11), (16) becomes:

$$\frac{\exp \beta V_i(+1, \phi_j)}{\exp(\beta V_i(+1, \phi_j)) + \exp(\beta V_i(-1, \phi_j))} = \frac{\exp[\beta/2(h + (1/pN)\sum_{j\in\nu_i} J_{ij}\phi_j)]}{\exp[\beta/2(-h - (1/pN)\sum_{j\in\nu_i} J_{ij}\phi_j)] + \exp[\beta/2(h + (1/pN)\sum_{j\in\nu_i} J_{ij}\phi_j)]} \tag{17}$$

The randomness of the conditional responses can be interpreted either from an econometric viewpoint, as a cross sectional heterogeneity in a sample population, or as an intrinsic randomness in behavior due to bounded rationality. When V_i only depends on ϕ_i this reduces to the standard logit discrete choice model (Anderson et al., 1992; Manski and McFadden, 1981). When choices are interdependent, Eqs. (16) or (17) represent the influence of social interactions via the *conditional* distribution $\mathbb{P}(\phi_i|\phi_j, j \in \nu_i)$ of actions given the social environment (Brock and Durlauf, 2000).

When $\beta \to \infty$, the agent chooses with probability one the alternative which maximizes social utility, completely disregarding individual tastes, resulting in a very similar behavior across individuals. By contrast the case $\beta = 0$ correspond to choosing alternatives with a uniform probability, resulting in a very heterogeneous cross section of individual behaviors. Hence the case where $\beta > 0$ interpolates between these two extreme cases: a large β indicates a behavior close to social utility maximization, a small β a "random" behavior. It will be therefore interesting to study how the properties of the model depend on β when β varies in $[0, \infty[$.

1.6. Objectives

Having described our model, we can now examine which quantities and properties are of interest:

(1) Equilibria: different notions of equilibrium, their existence and their properties are studied in Section 2.
(2) Peer group effects: what is the impact, at equilibrium, of the characteristics of an individual on her choice? In Section 2.4 we propose a simple way to quantify this feature and show that it allows a characterization of the equilibria of the model.
(3) Empirical content of the model: can the model conclusions be (in)validated empirically? How can the model parameters be estimated from survey type data on agents characteristics and actions?
(4) Effect of biases in individual preferences: how does a non-zero bias h affect equilibrium actions? Can equilibria be selected or modified by a policy maker providing incentives to orient actions? This question is studied in Section 4.

We will show that, when the number of agents N is large, many of these questions can be answered in a precise manner.

2. Equilibria

The interdependence of agents decisions $(\phi_i)_{i=1\ldots N}$ requires an additional criterion in order to specify the decisions in a consistent way (Blume and Durlauf, 2003; Horst and Scheinkman, 2006). Previous studies (Brock and Durlauf, 2000; Glaeser and Scheinkman, 2002; Horst and Scheinkman, 2006) consider a notion of static self-consistent equilibrium, based on self-consistency of the actions or beliefs. This approach, discussed in Section 2.1, does not describe how equilibrium is reached. In Section 2.2 we describe a dynamic process by which agents update their choices, consistently with their preferences.

This allows us to define dynamic equilibria, which are studied in Section 2.3. Dynamic equilibria turn out to be different, in general, from static equilibria.

2.1. Static self-consistent equilibria

In social interaction models discussed in the literature, the notion of equilibrium is often the following (Blume and Durlauf, 2003; Horst and Scheinkman, 2006): a configuration of actions $(\phi_i^*)_{i=1\ldots N}$ is a (static) equilibrium of the social interactions models defined by (3)–(13) if the action of each individual is the best response to the actions of others:

$$\forall i = 1 \ldots N, \quad \phi_i^* \in \underset{\phi_i \in S}{argmax}\, U_i(\phi_i, \{\phi_j^*, j \neq i\}) \tag{18}$$

In general $(\phi_i^*)_{i=1\ldots N}$ is a random variable. More precisely in the context of our model, a *self-consistent equilibrium* can be defined by a (joint) probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that under $\mathbb{P}$, (3)–(7)–(10) hold and (18) holds almost surely. $\mathbb{P}$ is interpreted as the joint law at equilibrium of the charactertistics, preferences and decisions of the agents. Conditions (3)–(7)–(10) simply mean that $\mathbb{P}$ gives a correct statistical description of the structure of social groups while condition (18) is a condition of self-consistency among the decisions of different agents.

Eq. (18) defines $(\phi_i^*)_{i=1\ldots N}$ as a fixed point of the (upper semi-continuous) correspondence:

$$(\phi_i)_{i=1\ldots N} \rightarrow \left(\underset{S}{argmax}\, U_i(.,\{\phi_j, j \neq i\})\right)_{i=1\ldots N} \tag{19}$$

The Kakutani fixed point theorem entails the existence of a self-consistent equilibrium.

However the mere existence of such equilibria does not tell us whether they can be reached, starting from an initial non-equilibrium configuration. In fact the computation of such equilibria would already require a certain sophistication on the part of the agents which may not be meaningful to assume in typical applications of social interaction models. Also note that the definition of self-consistent equilibria as the specification of a random variable requires the knowledge of the joint law of actions along with sources of randomness in the environment (preferences, characteristics, communication between individuals). From an empirical point of view, when we observe a given population it is not possible to distinguish between two such (self-consistent) equilibria which give the same conditional distributions for actions given the environment. Thus multiple equilibria may exist in the model but these equilibria may be observationally equivalent.

For these reasons we tackle the equilibrium problem through another angle: we first define a rule for updating agents decisions which is compatible with the conditional preferences describes in Section 1.4 and introduce the notion of dynamic equilibrium as a statistical distribution of actions which is invariant under the dynamics considered.

2.2. Dynamic updating of choices

A natural dynamic specification for the evolution of agents choices is a sequential updating scheme. However, since there is more than one agent involved and choices are interdependent, one should specify in which order the agents choices are to be updated. Since in a natural setting this order is random, one can consider that each agent updates his/her choice at random time intervals which, for example, are IID random variables with an exponential distribution. The updating times for different agents are therefore random and independent, such that, with probability one, choices of different agents will be updated at different times: when they take decisions, agents react to what they observe other agents in their peer group are doing. An agent reacts to the actions of other as opposed to his anticipations on actions of other (as in Brock and Durlauf, 2000). This should be contrasted with *s* ynchronous updating schemes in repeated games with strategic interactions where choices are determined simultaneously through in a non-cooperative equilibrium as in (18) at each time step. The perspective of social interaction models in large populations is not to model strategic interactions but collective behavior of non-strategic nature.

Formally, one can consider N IID Poisson processes $(\Gamma_i(t), i = 1 \ldots N)$ and define the decision times of agent i with the jump times $t_1^i, \ldots, t_k^i, \ldots$ of Γ_i. Such an approach is also used for instance in Blume and Durlauf (2003). Thus, the intervals between two updating times are IID exponentially distributed random variables, independent across agents.

Let us stress that there is no natural order in which agents should update their choices: this order is random. This should be contrasted with models of herd behavior (Bannerjee, 1992) in which individuals are *ordered* and update their choices in this predetermined order. Obviously in a realistic situation such an ordering is not available ex-ante.

At the updating time t_k^i, agent i observes the state of other agents $\phi_j(t), j \neq i$ and chooses $\phi_i(t_k^i) = +1$ with probability:

$$\frac{\exp \beta \sum_{j \in \nu_i} \phi_j J_{ij}}{\exp(\beta \sum_{j \in \nu_i} \phi_j J_{ij}) + \exp(-\beta \sum_{j \in \nu_i} \phi_j J_{ij})}. \tag{20}$$

When $\beta \to \infty$ this rule leads to choosing the outcome which maximizes the social utility $V(.;\phi_j, j \neq i)$. When $\beta = 0$, agents chooses randomly between alternatives regardless of what other agents are doing.[3]

2.3. Dynamic equilibria

With the dynamic evolution rules described above, the state vector $\phi(t) = (\phi_i(t), i = 1 \ldots N)$ is then a continuous-time Markov process with state space $\{-1, +1\}^N$. Given that N is typically a large number, only a statistical description of the state of the system i.e. the distribution of $\phi(t)$ may be available to an observer: for example, statistics may be obtained from survey data or some aggregate variables may be observed. Two states giving rise to the same statistical observations are therefore not distinguishable in practice.

An equilibrium state is therefore defined as a state in which cross-sectional properties of $\phi(t)$ do not change with time. From a mathematical point of view, this corresponds to an invariant distribution of the Markov process $\phi(t)$. We therefore define a *dynamic equilibrium* as an invariant distribution of the Markov process $\phi(t)$. Note that, in contrast to the notion of self-consistent equilibrium defined in 2.1, the notion of dynamic equilibrium corresponds to the specification of a (marginal) probability distribution on actions $(\phi_i)_{i=1\ldots N}$ only, as opposed to a joint distribution on actions of agents and their environment. This distinction is important since the marginal distribution of actions $(\phi_i)_{i=1\ldots N}$ can be estimated by observing (sampling) agents behavior therefore this notion of equilibrium does have empirical content. The following theorem shows the existence and uniqueness of dynamic equilibrium and provides a description of the distribution of actions in equilibrium:

Proposition 2.1 (Dynamic equilibrium). *The Markov process of agents' choices $\phi(t) = (\phi_i(t))_{i=1,\ldots N}$ evolving under the rules described in* Section 2.2 *has a unique stationary distribution given by*

$$\forall \phi \in \{-1, 1\}^N \quad \mu_{N,\beta}(\phi) := \frac{e^{-\beta H_{N,\xi}(\phi)}}{\sum_{\sigma \in \{-1,+1\}^N} e^{-\beta H_{N,\xi}(\sigma)}}. \tag{21}$$

where $H_{N,\xi} : \{-1, 1\}^N \to \mathbb{R}$ is defined by

$$H_{N,\xi}(\Phi) := -\frac{1}{2pN} \sum_{i,j=1}^{N} J_{ij} \phi_i \phi_j \tag{22}$$

Then for large t, any fixed N, and any initial distribution of $\phi(0) \in \{-1, +1\}^N$ the distribution of $\phi(t)$ converges to the $\mu_{N,\beta}$.

Proof. The jump times of the independent Poisson processes $(\Gamma_i)_{i=1\ldots N}$ are almost surely distinct and can be arranged into a strictly increasing sequence $\tau_1 \leq \ldots \leq \tau_n, \ldots$ with $\tau_n \to \infty$ almost surely. By conditioning the Markov process on the updating times $(\tau_n)_{n \geq 0}$ and defining $\phi(n) = \phi(\tau_n)$ we obtain a Markov chain $(\phi(n))_{n=0,1,\ldots}$ with transition probabilities

$$P(\phi_i(n+1) = \phi | \phi_j(n), j \in \nu_i) = \frac{e^{\beta \sum_{j \in \nu_i} \phi \phi_j J_{ij}}}{e^{\beta \sum_{j \in \nu_i} \phi_j J_{ij}} + e^{-\beta \sum_{j \in \nu_i} \phi_j J_{ij}}}. \tag{23}$$

It is easily checked that the transition probabilities (23) of the Markov process are reversible with respect to the distribution $\mu_{N,\beta}$: denoting the transition operator of the Markov chain by T:

$$T.\mu_{N,\beta} = \mu_{N,\beta}.T = \mu_{N,\beta}$$

Hence $\mu_{N,\beta}$ is a stationary distribution for the Markov chain. It remains to check that the Markov chain is irreducible. Since the state space is finite, it is sufficient to check that the transition probability between two configurations ϕ and ϕ' differing by a single coordinate is strictly positive. This indeed the case since the transition probabilities in (23) are strictly positive. Hence $\mu_{N,\beta}$ is the only stationary measure. Also since the Markov process is ergodic it will converge to its stationary measure when $t \to \infty$ starting from any initial distribution. □

The above result shows that the study of the long run behavior of $\phi(t)$ boils down to the study of $\mu_{N,\beta}$.

Note also that, in a given population defined by the characteristics (ξ_i^α), the dynamic distribution will depend on the population characteristics: the equilibrium distribution $\mu_{N,\beta}$ is a *random* measure. Hence, a priori, two different populations following the same dynamic rules will in general have different dynamic equilibria. It is therefore interesting to see what an equilibrium looks like in a *typical* population. Are then any properties of $\mu_{N,\beta}$ which hold for *any* typical population (i.e. almost surely with respect to population characteristics)? This point will be further discussed in Section 3.

[3] This was first remarked in the context of neural networks by Hopfield (1982) and was the starting point of the statistical mechanics of neural networks (Mezard et al., 1988).

Remark 2.2 *(Relation with evolutionary game theory).* As noted in the introduction, such stochastic choice revision schemes have also been studied in the literature on evolutionary game theory (Baron et al., 2004). The approach in this literature is to study the limit of the invariant distribution as the noise level goes to zero i.e. $\mu_N^* = \lim_{\beta\to\infty}\mu_{N,\beta}$. The states in the support of μ_N^* are interpreted as stochastically stable states (Foster and Young, 1990) and in many cases constitute a subset of Nash equilibria of the unperturbed game with $\beta = \infty$. Noise is thus used as an equilibrium selection mechanism. By contrast, in the present case "noise" is not asymptotically reduced to zero but constitutes an essential component of the description of individual behavior. In Section 3, we will study the behavior of dynamics equilibria when β is *fixed* and the number of agents N becomes large.

2.4. Correlation between characteristics and decisions

The equilibrium measure defined above gives a statistical description of the system in terms of the states (choices) of individual agents $\phi = (\phi_i, i = 1\ldots N)$. An important question which is considered in empirical studies of population behavior is the correlation between certain individual characteristics and the outcome of individual decisions. In the case of survey data, one can quantify this correlation by computing the sample covariance between the decision variable ϕ_i and each of the M characteristics ξ_i^α, $\alpha = 1\ldots M$:

$$m_{N,\xi}^\alpha(\phi) = \frac{1}{N}\sum_{i=1}^{N}\xi_i^\alpha\phi_i \tag{24}$$

This quantity, which can be interpreted as the population covariance between the α-th characteristic and the individual decisions ϕ_i, also appears in the cross-sectional regression of ϕ_i on ξ_i^α. The subscript N, ξ indicates the dependence on the population size N and the population characteristics ξ. $m_{N,\xi}^\alpha$ measures the influence of the α-th characteristic on the decision of the agents in the population: for example, if a certain social group (whose individuals are characterized by $\xi_i^1 = 1$) has a consumption pattern which leads members to choose $\phi_i = +1$ with high probability then $m_{N,\xi}^1$ will be very close to the maximal value. These quantities can therefore be used to quantify "peer group" effects.

In addition to their obvious empirical interest, the sample correlations m^α, $\alpha = 1\ldots M$ play an important theoretical role in the case where $p = 1$ (i.e. when the communication structure is described by a completely connected graph). In this case one can rewrite $H_{N,\xi}$ (and hence also $\mu_{N,\beta}$) as a function of the equilibrium correlations *only*:

$$\mu_{N,\beta}(\phi) := \frac{\exp(-(\beta N/2)\|m_{N,\xi}(\phi)\|^2)}{\sum_{\sigma\in\{-1,+1\}^N}\exp(-(\beta N/2)\|m_{N,\xi}(\sigma)\|^2)} \tag{25}$$

Here $m_{N,\xi}(\phi) := (m_{N,\xi}^\alpha(\phi), \alpha = 1, \ldots M)$, $< \cdot, \cdot >$ denotes the inner product in $\mathbb{R}^M$ and $|\cdot|$ the Euclidean norm. Therefore in the case where $p = 1$, the equilibrium correlations $m_{N,\xi}(\beta) = (m^\alpha, \alpha = 1\ldots M)$ are the (only) relevant variables for the description of dynamic equilibrium. This is interesting since these quantities can be estimated in the context of survey analysis and are used to interpret survey results (Borjas, 1992; Evans et al., 1992; Glaeser et al., 1996; Jones, 1994). This can be restated by saying that these correlation coefficients are *sufficient statistics* for the description of the equilibrium distribution so the procedure of using them as descriptive variables is indeed the correct way to proceed. Note however that this is not true anymore if $p < 1$ i.e. if communication is imperfect. In the general case one needs more than the knowledge of macroeconomic variables m^α, $\alpha = 1\ldots M$ to pinpoint the equilibrium distribution: one also needs to know something about the social communication structure (ε_{ij}). However the equilibrium correlations m^α, $\alpha = 1\ldots M$ are still interesting quantities since they still represent the impact of characteristics on individual decisions.

Note that the quantities $(m^\alpha)_{\alpha=1\ldots M}$ are random variables whose equilibrium values depend on the population characteristics (ξ_i^α). The vector $m_{N,\xi}$ is thus a $\mathbb{R}^M$-valued random vector, whose distribution we denote by $\mathcal{Q}_{N,\xi,\beta}$:

$$\mathcal{Q}_{N,\xi,\beta}(A) = \mu_{N,\beta}(m_{N,\xi}\in A).$$

By construction $m_{N,\xi}^\alpha$ is an aggregate quantity: it is defined as a population average. A natural question is then to ask whether it obeys a "law of large numbers": does it converge to a well defined value when N is large? If yes, what is this value and how does it depend on the population characteristics ξ on one hand and on the 'heterogeneity' parameter β on the other hand? The answer is not obvious since the individual decisions ϕ_i are far from being IID: social interactions induce a complex dependence structure among them. These questions can be answered by the study of the limit behavior of $\mathcal{Q}_{N,\xi,\beta}$ when $N \to \infty$, as shown in the next section.

3. Equilibria in large populations

The (random) variables $(m^\alpha)_{\alpha=1\ldots M}$ quantify peer group effects and may be interpreted as population averages or cross sectional moments. A natural question is then to investigate their behavior when the population size is large. Do they obey a "law of large numbers" i.e. do they converge to a deterministic limit? If yes, then this limit can be used to describe the effect of conformism on equilibrium in a large population.

Studying the large population limit has another interesting by-product: as we shall see below, the properties of dynamic equilibria in this limit do not depend on the details of the population characteristics, which is very important from an empirical point of view.

For a given population size N we have a random configuration $\phi^N = (\phi_i^N, i = 1 \ldots N)$, whose distribution is given by $\mu_{N,\beta}(\phi)$ and the "equilibrium correlations" m^α whose distribution is $\mathcal{Q}_{N,\xi,\beta}$. There are various ways to define and study the convergence of such a sequence when the population size N increases. In a framework of statistical modeling of agents behavior, the only relevant quantities to compare with observations are aggregate macroeconomic variables which correspond to population averages. We are therefore interested in expectations of various quantities under $\mathcal{Q}_{N,\xi,\beta}$ and the limits of these expectations will define the behavior of the corresponding aggregate variable. The convergence of expectations of (bounded continuous) functions corresponds to the mathematical notion of *weak convergence* of measures.

We study in this section the properties of the model when the population size is large. First, we will show that in a large population, the sample correlations m^α can only assume a finite number of values. These values can in turn be computed from the heterogeneity parameter β (defined in Section 1). For large enough values of β, this will lead us to conclude to the existence of multiple equilibria while in heterogeneous populations (characterized by small β) the values of equilibrium correlations will be equal to zero. This result will then be used to propose an estimator for the parameter β and show that it is consistent for almost any population as the size of the population grows.

3.1. Large population behavior of equilibrium correlations

Assume without loss of generality that

$$1/2 \le \pi^1 = \pi^2 = \ldots = \pi^k < \pi^{k+1} \le \ldots \le \pi^M \tag{26}$$

In mathematical terms, we will show that when $N \to \infty$, the equilibrium correlations $m_{N,\xi} = (m^\alpha_{N,\xi}, \alpha = 1 \ldots M)$ will take their values in the finite set with $2k$ elements:

$$\Lambda = \{\pm z(\beta) e^\alpha, \alpha = 1 \ldots k\} \tag{27}$$

where $z(\beta)$ is the largest solution of:

$$z = 2\pi^1(1-\pi^1)[\tanh(2\beta(1-\pi^1)z) + \tanh(2\beta\pi^1 z)] \tag{28}$$

Denote by Λ^ϵ the ϵ-neighborhood of Λ:

$$\Lambda^\epsilon = \bigcup_{x \in \Lambda} B(x, \epsilon) \tag{29}$$

Our main result can be stated as follows:

Proposition 3.1. *Let $\mathcal{Q}_{N,\xi,\beta}$ be the distribution (on $\mathbb{R}^M$) of $m_{N,\xi}(\phi)$ under the equilibrium measure $\mu_{N,\beta}$. Define*

$$\beta_1 = \frac{1}{4\pi^1(1-\pi^1)} \tag{30}$$

Then, as $pN \to \infty$

(1) *For $\beta \le \beta_1$ and for almost all population characteristics ξ, $\mathcal{Q}_{N,\xi,\beta}$ converges weakly to the constant zero as $N \to \infty$ i.e. for any bounded continuous function $f : \mathbb{R}^M \mapsto \mathbb{R}$,*

$$E^{\mu_{N,\beta}}[f(m_{N,\xi}(\phi))] \overset{N\to\infty}{\to} f(0) \qquad \text{almost-surely.} \tag{31}$$

(2) *For $\beta > \beta_1$ and for almost all population characteristics ξ, the support of $\mathcal{Q}_{N,\xi,\beta}$ becomes concentrated on the finite set Λ:*

$$\forall \epsilon > 0, A \cap \Lambda^\epsilon_M = \varnothing \Rightarrow (\mathcal{Q}_{N,\xi,\beta}(A) \to 0). \tag{32}$$

The proof of this result is given in Appendix B

In the special case where $\pi^\alpha = \pi$ does not depend on α (i.e. where the characteristics are IID) and $p = 1$ (completely connected graph) the model described above is mathematically equivalent to the Hopfield model of neural networks (Pastur and Figotin, 1977; Hopfield, 1982; Mezard et al., 1988) and the corresponding limit theorem was first proved in Bovier et al. (1994) (see also Comets, 2002; Pastur and Figotin, 1977). The case where $\pi^\alpha = \pi$, $p < 1$ was studied in Bovier and Gayrard (1993).

Let us now comment on the meaning of these results. First, note that by definition dynamic equilibrium for a finite population corresponds to a *random* configuration of individual actions with distribution $\mu_{N,\beta}$, which depends on the population characteristics. However, the result above shows that when the population is large, we will observe a single *deterministic* configuration if $\beta < \beta_1$. The "almost-sure" nature of the convergence means that all these results are valid for any "typical"

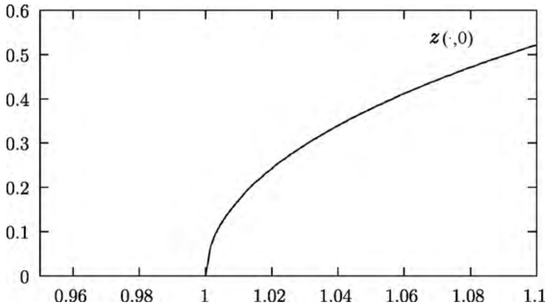

Fig. 1. Equilibrium correlation as a function of the heterogeneity parameter β.

population and are therefore robust to the details of population characteristics."Typical" here refers to any set of population characteristics which has non-zero probability of occurring given the distribution of individual characteristics described in Section 1.1. Also, we see that a qualitative change in the behavior of the equilibrium properties occurs when the heterogeneity parameter β crosses the critical value β_1. For $\beta < \beta_1$, there is no cross-sectional correlation between the agents behavior and their characteristics: individual diversity dominates the outcome. It was already expected that the lower β is, the less the social interactions will tend to overrule the heterogeneity of individual preferences. However the above result shows that there is a t hreshold value of β below which social interactions dominate individual heterogeneity to the point of altering the nature and number of equilibria. This abrupt change of behavior at $\beta = \beta_1$ is usually called a 'phase transition' and is observed in many similar models with social interactions (Föllmer, 1974; Brock and Durlauf, 2000; Glaeser et al., 1996). In terms of the equilibrium distribution, this means that when individual tastes are heterogeneous enough ($\beta < \beta_1$) the distribution of choices is close to being uniform, just as in a random utility model with IID utilities and *no* social interaction.

When $\beta > \beta_1$, more interesting phenomena happen. First, the equilibrium correlations between choices and characteristics will take a non zero value. More precisely, the above results mean that only those characteristics which divide the population most evenly (i.e. with π^α closest to 1/2) will play a role in determining the long run equilibrium. Let us call them d ominant characteristics in the sequel. According to the notation defined in Eq. (26), these are the characteristics labeled ξ^α, $\alpha = 1 \ldots k$. Of course the generic situation is the one where $\pi^1 < \pi^2$ so $k = 1$. Moreover, out of these dominant characteristics, only *one* of them (say, α_0) will be correlated with the agents behavior at equilibrium (the others having zero correlation with the behavior). This characteristic α_0 can be any of the k dominant characteristics $\alpha = 1 \ldots k$, each choice leading to a different (deterministic) configuration: for $\beta > \beta_1$ we therefore have at least $2k$ possible decision configurations which can occur at equilibrium and an observer of the system will conclude that there are "multiple equilibria". The characteristic $\xi_i^{\alpha_0}$ will then have a non-zero correlation with the agents behavior ϕ_i. However, the value of this equilibrium correlation is not any arbitrary number in $[-1, 1]$: it can only take the two possible values $\pm z(\beta)$. This value of the equilibrium correlation only depends on β: this dependence is shown for $\pi^1 = 1/2$ in Fig. 1.

3.2. *Measuring the heterogeneity of individual preferences*

The results above have interesting implications for the empirical content of the model. Apart from the population characteristics ξ_i^α, β is the only parameter in this model and is not, in principle, an observable quantity. In a discrete choice model *without* social interactions, the individual choices ϕ_i are regarded as IID variables with a distribution in which β is the (only) parameter (Anderson et al., 1992) and estimating β from a sample (ϕ_i, $i = 1 \ldots N$) is a classical parametric estimation problem. When social interactions are introduced, (ϕ_i, $i = 1 \ldots N$) are no longer IID variables: their dependence is especially strong if $\beta > \beta_1$. Also, the equilibrium distribution (21) of ϕ not only depends on β but also on the population characteristics ξ. So it does not seem possible at first glance to infer β from aggregate, survey-type data on the population (Fig. 2). The following result shows that β can still be estimated via an *aggregate* quantity which, if the population is large, becomes population-independent:

Proposition 3.2. *Denote by $z(\beta)$ the largest solution to* Eq. (28) *and by $z^{-1} : [0, \infty[\to [\beta_1, \infty[$* its inverse function. For $m_{N,\xi}^\alpha > 0$ *define*

$$\hat{\beta}_{N,\xi}^\alpha := z^{-1}(|m_{N,\xi}^\alpha|) = z^{-1}\left(\left|\frac{1}{N}\sum_{i=1}^N \phi_i \xi_i^\alpha\right|\right) \tag{33}$$

$$\hat{\beta}_{N,\xi} := \sup(\overset{\alpha}{\hat{\beta}}_{N,\xi}, \quad \alpha = 1 \ldots M) \tag{34}$$

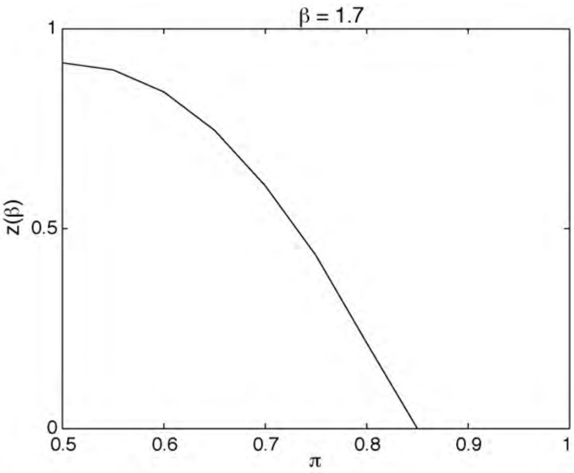

Fig. 2. The equilibrium correlation $z(\beta)$ for $\beta = 1.7$ as a function of π^1.

Then if $\beta > \beta_1$, $\hat{\beta}_N$ is a consistent estimator of β for almost all ξ:

$$\hat{\beta}^{\alpha}_{N,\xi} \underset{N\to\infty}{\to} \beta. \tag{35}$$

Proof. Let $\beta > \beta_1$. Then by Theorem 3.1 for any typical population (i.e. almost surely in ξ), the measure $\mathcal{Q}_{N,\xi,\beta}$ concentrates on the finite set Λ defined in (27). But on Λ, either $|m^\alpha| = |z(\beta)|$ or $m^\alpha = 0$ with $|m^\alpha| = |z(\beta)|$ for at least one component, say α_0. Then for $\alpha \neq \alpha_0$,

$$\left|\frac{1}{N}\sum_{i=1}^{N}\phi_i\xi_i^\alpha\right| \underset{N\to\infty}{\to} 0 \quad \text{and} \quad \left|\frac{1}{N}\sum_{i=1}^{N}\phi_i\xi_i^{\alpha_0}\right| \underset{N\to\infty}{\to} z(\beta)$$

almost surely in ξ. So $z(\hat{\beta}^{\alpha}_{N,\xi}) \to z(\beta)$ almost surely in ξ. Since $z :]\beta_1, \infty[\to]0, \infty[[$ is one to one we obtain the assertion of the theorem. □

Again, the almost sure character with respect to ξ of the convergence in N means that the estimator is consistent for any typical population (Fig. 3). In order to assess the asymptotic distribution of the $\hat{\beta}_N$ defined above, one can naturally ask whether there is a central limit theorem associated to the law of large numbers given in 3.1. In the case $p = 1, \pi^\alpha = 1/2$, such a result has been shown by Gentz and Löwe (1999) for $\beta > \beta_1$. However the variance diverges as $\beta \to \beta_1$: the fluctuations around equilibrium can be shown (Gentz and Löwe, 1999) to be of order

$$c_\beta\sqrt{N} := \frac{1 - z(\beta)^2}{1 - \beta(1 - z(\beta)^2)}\sqrt{N}.$$

That is, while on average $z(\beta)N$ agents will tend to behave according to the limit distribution derived in Theorem 3.1, around $c_\beta\sqrt{N}$ agents will deviate from this behavior. The graph of the function

$$\beta \mapsto \frac{1 - z(\beta)^2}{1 - \beta(1 - z(\beta)^2)}, \tag{36}$$

is shown in Fig. 4: the fluctuations become larger and larger, as β gets closer to the critical value β_1 [4]: which means that when $\beta \simeq \beta_1$ it is not possible to identify accurately β from the aggregate covariances m^α. This breakdown of the central limit theorem close to the phase transition is a typical property of systems with multiple equilibria.

4. Biased preferences and incentives

As shown in Section 2.3, starting from any initial state the system described above eventually self-organizes into a statistical equilibrium described above. However in the case where social pressure does not dominate the effect of heterogeneity ($\beta > \beta_1$), Theorem 3.1 shows that multiple equilibria can occur.

[4] In fact in the case $p = 1$, at $\beta = \beta_1$, the fluctuations are no longer of order $\sqrt{N}$ but of order $N^{3/4}$.

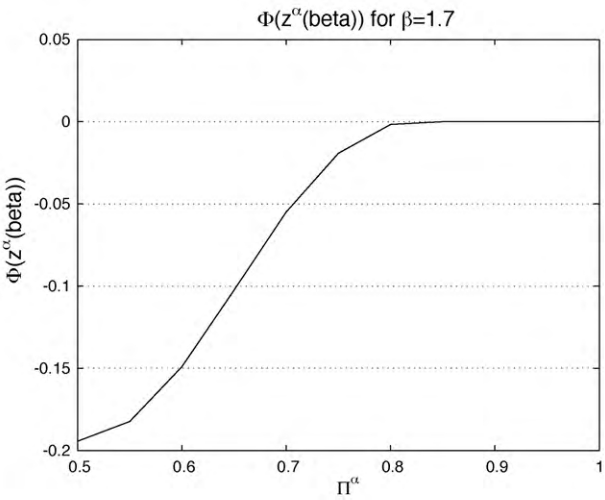

Fig. 3. The value of the minimum of $\Phi(.)$ as a function of π^α for $\beta = 1.7$.

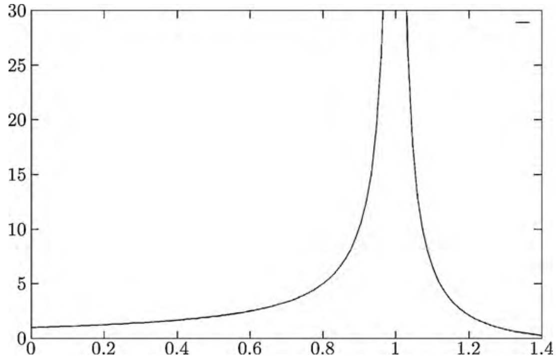

Fig. 4. Standard deviation of $\hat{\beta}_N$ as a function of β (Eq. (36)).

Given that the model exhibits different regimes of behavior, it is of interest to see whether a policy maker can influence the emergence of a particular type of behavior by providing appropriate incentives to the agents. Consider a policy maker providing *incentives* to agents in order to orient their actions: these incentives may be modelled by an exogenous term introduced in the utility function, increasing the utility of one of the choices. In our binary choice model, this may be modelled as an exogenous additive term in the utility function:

$$U_i^h(\phi_i; \{\phi_j, j \neq i\}) = U_i(\phi_i; \{\phi_j, j \neq i\}) + h\phi_i\xi_i^{\alpha_0} \tag{37}$$

Thus when $h > 0$, the incentive policy provides a gain in utility of $h(\xi_+^{\alpha_0} - \xi_-^{\alpha_0})$ to an agent with characteristic $\xi_i^{\alpha_0} = \xi_+^{\alpha_0}$ for choosing $\phi_i = +1$: the policy is directed towards individuals possessing the characteristic α_0.

Taking into account the effect of incentives, it is easy to see that the existence and uniqueness of dynamic equilibrium given in Theorem 2.1 still hold. However, the equilibrium distribution is now given by

$$\mu_{N,\xi,\beta}(\phi) := \frac{e^{-\beta H_{N,\xi,h}(\phi)}}{\sum_{\sigma \in \{-1,+1\}^N} e^{-\beta H_{N,\xi,h}(\sigma)}} \tag{38}$$

where $H_{N,\xi,h}$

$$H_{N,\xi,h}(\phi) := -\frac{1}{2N}\sum_{i,j=1}^{N}\sum_{\alpha=1}^{M}\xi_i^\alpha\xi_j^\alpha\phi_i\phi_j - h\sum_{i=1}^{N}\xi_i^{\alpha_0}\phi_i. \tag{39}$$

By comparing the new equilibrium to the one in absence of any incentive policy, one may now discuss how the policy affects the equilibrium relation between actions and individual characteristics, as measured by m^α. Indeed, following the lines of the proof of Proposition 3.1 one can show the following:

Proposition 4.1. *Assume that*

$$Np \underset{N\to\infty}{\to} \infty \tag{40}$$

Let $e^{\alpha_0} \in \mathbb{R}^M$ be the unit vector in direction α_0. Then, as $N \to \infty$ for any β and any $h \neq 0$, the equilibrium correlations $m_{N,\xi}(\beta)$ converge in distribution to $z(\beta, h)e^{\alpha_0}$ with probability one with respect to the individual characteristics ξ:

$$\mathcal{Q}_{N,\xi,\beta} \Rightarrow \delta_{z(\beta,h)e^{\alpha_0}}. \tag{41}$$

where $z(\beta, h)$ denotes the unique solution of

$$z = 2\pi^{\alpha_0}(1 - \pi^{\alpha_0})[\tanh(2\beta\pi^{\alpha_0}z) + \tanh(2\beta(1 - \pi^{\alpha_0})z)] \tag{42}$$

having the same sign as h.

The proof, similar to Theorem 3.1 can be obtained following the lines of Bovier and Gayrard (1993); Bovier et al. (1994) and is omitted for the sake of brevity.

This result shows two phenomena. First, in the case where individual preferences are not very heterogeneous ($\beta > \beta_1$, see Theorem 3.1) an incentive policy can *select* one equilibrium from many. Second,it can also generate new equilibria in the case $\beta < \beta_1$ where precedingly equilibrium was unique. These observations point to social multiplier effects, already observed in other social interaction models, where providing (small) incentives to agents can create effects on a macro-economic scale.

In the more general case where the incentives $h = (h^\alpha)_{\alpha=1\ldots M}$ may affect various social groups in different ways, one can show that when N is large the policy defined by h and $h^{\max}e^{\max}$ (where $h^{\max}$ is the component of h with the largest modulus $|h^{\max}| = \max_\alpha |h^\alpha|$ and $e^{\max}$ is the corresponding unit vector) give the same result: the equilibrium distribution is "peaked" in the direction where the incentives are the largest and configurations in other directions have a small probability to occur. This means that efficient incentive policies are those directed towards a single social group. For example, if the cost of implementing a policy $h = (h^\alpha)$ is proportional to the incentive $|h^\alpha|$ it provides, then the same overall result may be achieved by replacing h by $h_{\max}e_{\max}$. Finally, the incentive policy also modifies the equilibrium correlations between individual characteristics and decisions: their value is now given by $z(\beta, h^{\alpha_0})$, solution of Eq. (42). Their effect therefore leads to empirically observable consequences in terms of cross-sectional regression of actions on characteristics.

5. Conclusion

We have presented in this work a model of binary choice social interactions in which agents actions influence other agents preferences in way which depends on the similarity of their characteristics. Our model allows for a random (population-dependent) structure of interactions and allows both for positive interactions of imitative nature ("strategic complementarity") and for *negative* interactions, a feature that is not present in previously studied models.

In a population where characteristics are randomly distributed across agents and where the communication between two agents is also allowed to be random, we have shown that the repeated choice revision process admits a unique dynamic equilibrium, which depends on the population characteristics. These equilibria are described through the aggregate relation between characteristics of individuals and their decisions, which we measure by the corresponding sample covariances. When the population is large, these sample covariances can only take a finite number of values, which are sample independent: depending on the degree of heterogeneity of individual preferences, this leads us to a single or multiple equilibria, the set of which is the *same* for almost all populations.

When there are multiple equilibria, and notwithstanding the dependence of individual behaviors, we have shown that the aggregate observations obey a law of large numbers which may be used to infer parameters describing the heterogeneity of individual tastes from observations of actions.

Let us now conclude by indicating some questions of further interest related to our model.

The existing literature on social interaction models focuses on equilibrium behavior, and we have followed the same path in the present work. In Sections 2.3 and 3 we studied equilibrium properties of the model and showed that the corresponding dynamics equilibria are asymptotically reached from any initial configuration: they characterize the long run behavior of the system. This convergence takes place with probability one: any typical initial configuration will converge to the dynamic equilibria studied above. However, we have not characterized the rate of convergence to such equilibria. How long does an external observer have to sample observations from the system in order for his/her observations to be comparable to a random draw from the equilibrium distribution? The answer to this question turns out to be non-trivial: in fact, one can show examples of our model in which the system gets trapped in a non-equilibrium configuration for a very long time before reaching equilibria. Such "slow dynamics" is in fact characteristic of many dynamical systems exhibiting structural (as opposed to dynamical) randomness (Mezard et al., 1988); see (Aoki, 1997) for some examples motivated by economics. If equilibrium takes a long time to reach as suggested by the above remarks, then the study of the *short-run* behavior of such models becomes of independent interest. In fact, social interaction models have often been suggested to describe fads and fashions which are, by essence, transitory phenomena (Schelling, 1973; Simmel, 1904). Whereas the equilibrium behavior studied in Section 3 is shown to be the same for any typical population, the same remark does not hold for short-run behavior in general. It is of interest to see what can be inferred for such transitory behavior of aggregate variables. Further research on these issues in the context of social interaction models is needed to better understand the empirical content of these models.

Another issue of interest is to explore the influence of more realistic communication structures between agents which resemble those observed in social networks, such as small world networks (Cont and Tanimura, 2008), or network structures determined endogenously, as a result of the agents decision process (Jackson and Rogers, 2005).

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Appendix A. Proofs of lemmas

Lemma A.1. *Consider the function*

$$\Phi_{\xi,\beta}(z) = \frac{\beta\|z\|^2}{2} - E\log\cosh(\beta\langle\xi_i, z\rangle) \tag{43}$$

Then the global minima of $\Phi_{\xi,\beta}$ are of the form: $u(\beta)e^{\alpha}$ where (e^{α}, $\alpha = 1 \ldots M$) is the canonical basis of $\mathbb{R}^M$ and $u \in \mathbb{R}$ is a solution of Eq. (58).

Proof. Let us prove this lemma just for the simplest interesting case that $M = 2$ and $\pi^1 = \pi^2 = 1/2$. The general proof is very similar. Obviously all critical points of $\Phi_{\xi,\beta}$ have to fulfill

$$\forall\alpha = 1\ldots M, \quad z^{\alpha} = E[\xi_i^{\alpha}\tanh(\beta < \xi_i, z >)] \tag{44}$$

which in our case reads as

$$\begin{aligned} z^1 &= \frac{1}{2}[\tanh(\beta(z_1+z_2)) + \tanh(\beta(z_1-z_2))] \text{ and} \\ z^2 &= \frac{1}{2}[\tanh(\beta(z_2+z_1)) + \tanh(\beta(z_2-z_1))] \end{aligned} \tag{45}$$

Obviously $u(\beta)e^{\alpha}$, $\alpha = 1, 2$ where $u \in R$ is a solution of Eq. (58) is a solution of (45). Note that for β below the critical parameter value $\beta = 1$ this is the only solution of (45). By substracting the two equations one obtains

$$y := z_1 - z_2 = \tanh(\beta y) \tag{46}$$

Hence y solves (58) implying that there is $0 \le \gamma \le 1$ such that $z_1 = \gamma y$ and $z_2 = (1-\gamma)y$. Studying the function

$$\gamma \mapsto \Phi_{\xi,\beta}(z) \qquad \gamma \in [0, 1]$$

where $\beta > 1, z = (z_1, z_2), z_1 = \gamma y, z_2 = (1-\gamma)y$, and finally γ is the largest solution of (58), one sees that it becomes maximal for $\gamma = 0$ and $\gamma = 1$. □

Lemma A.2. *Let $z^*(\beta)$ denote the largest solution of the* Eq. (58). *Let $1/2 \le p \le 1$ denote the probability*

$$p := \mathbb{P}(\xi_i^{\alpha} > 0)$$

Then for any $\beta > 0$

$$p \mapsto z^*(\beta) =: z_p^*(\beta)$$

is decreasing. If β is such that $z^*(\beta)$ has more than one solution for $1/2 \le p_1 < p_2$ then even

$$z_{p_2}^*(\beta) < z_{p_1}^*(\beta).$$

Proof. It is easily verified by Taylor expansion that for any $\beta > 0, z > 0$, and any $1/2 \le p \le 1$ it holds

$$p \mapsto 2p(1-p)\left(\tanh(2\beta pz) + \tanh(2\beta(1-p)pz)\right)$$

is strictly decreasing. As $z_p^*(\beta)$ is defined by to be the largest (and therefore non-negative) solution of the fixed point equation

$$z = 2p(1-p)\left(\tanh(2\beta pz) + \tanh(2\beta(1-p)pz)\right)$$

this proves the result. □

Lemma A.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be the convex, increasing and twice differentiable function $g(x) = \log(cosh(x))$. Then for every fixed $x > 0$ the function*

$$H(p) := pg(2(1-p)x) + (1-p)g(2px) \tag{47}$$

is strictly increasing on the open interval $(0, 1/2)$.

Proof. By scaling it suffices to prove the result for $x = 1/2$. Now

$$\frac{d}{dp}H(p) = \log\cosh(1-p) - \log\cosh(p) + (1-p)\tanh(p) - p\tanh(1-p). \tag{48}$$

One verifies that $(1-p)\tanh(p) - p\tanh(1-p) \geq 0$ for $0 \leq p \leq 1/2$ and, of course also $\log\cosh(1-p) - \log\cosh(p) > 0$ for $0 \leq p < 1/2$. Hence

$$\frac{d}{dp}H(p) > 0$$

for $0 \leq p < 1/2$ proving the result. □

Lemma A.4. *Let $z_p^*(\beta)$ denote the largest solution of the* Eq. (58). *Let $1/2 \leq p \leq 1$ denote the probability*

$$p := P(\xi_i^\alpha > 0)$$

Then for any $\beta > 0$ the function

$$p \mapsto \Phi_{\xi,\beta,p}(z_p^*(\beta)) := \Phi_{\xi,\beta}(z_p^*(\beta))$$

is increasing. If $\beta > 1$ it is even strictly increasing.

Proof. By the above lemma the function $p \mapsto pg(2(1-p)x) + (1-p)g(2px)$ where $g(x) = \log(cosh(x))$ is strictly increasing on $(0, 1/2)$ for every fixed x. By symmetry it is *strictly decreasing* on $(1/2, 1)$. Now take $1/2 \leq p < q \leq 1$. If β is large enough the corresponding Eqs. (58) have more than the trivial solution $z = 0$. Let the largest (positive) solutions of (58) then be denoted by $z_p^*(\beta)$ and $z_q^*(\beta)$, respectively. Then

$$p' \mapsto p'g(2\beta(1-p')z_q^*(\beta)) + (1-p')g(2\beta p' z_q^*(\beta))$$

is decreasing, and therefore

$$p \mapsto \Phi_{\xi,\beta,p}(z_q^*(\beta))$$

is *strictly increasing*. Thus

$$\Phi_{\xi,\beta,p}(z_p^*(\beta)) \leq \Phi_{\xi,\beta,p}(z_q^*(\beta)) < \Phi_{\xi,\beta,q}(z_q^*(\beta)) \tag{49}$$

proving the lemma. □

Appendix B. Proof of Proposition 3.1

Proof. The proof will be given in two stages. First, we will indicate how the results are obtained in the case where $p = 1$ (i.e. the underlying communication structure is a completely connected graph). In the second part, using an inequality given in Bovier and Gayrard (1993), we will show how the results can be extended to the case where the communication structure is random.

(1) Case where $p = 1$ (completely connected graph):
The main idea in the proof of Theorem 3.1 consists in using the *Hubbard-Stratonovich* transformation (see Mezard et al., 1988). Define $\tilde{\mathcal{Q}}_{N,\xi,\beta}$, defined as the convolution of $\mathcal{Q}_{N,\xi,\beta}$ with a Gaussian measure in $\mathbb{R}^M$ with expectation 0 and covariance matrix $(\beta N)^{-1} Id$ (where Id denotes the identity matrix):

$$\tilde{\mathcal{Q}}_{N,\xi,\beta} := \mathcal{Q}_{N,\xi,\beta} * \mathcal{N}(0, (\beta N)^{-1} Id) \tag{50}$$

Since $\mathcal{N}(0, (\beta N)^{-1} Id)$ converges weakly to zero it is easily seen that $\tilde{\mathcal{Q}}_{N,\xi,\beta}$ and $\mathcal{Q}_{N,\xi,\beta}$ have the same weak limits. Moreover, a standard calculation involving the "Babylonian trick":

$$e^{a^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + ax}\, dx \tag{51}$$

yields that $\tilde{\mathcal{Q}}_{N,\xi,\beta}$ has a density with respect to M-dimensional Lebesgue λ^M of the form

$$\frac{d\tilde{\mathcal{Q}}_{N,\xi,\beta}(z)}{d\lambda^M(z)} = \frac{e^{-N\Phi_{N,\xi,\beta}(z)}}{\tilde{Z}_{N,\xi,\beta}}, \tag{52}$$

where

$$\tilde{Z}_{N,\xi,\beta} := \int e^{-N\Phi_{N,\xi,\beta}(z)} d\lambda^M(z) \tag{53}$$

$$\Phi_{N,\xi,\beta}(z) := \frac{\beta\|z\|^2}{2} - \frac{1}{N}\sum_{i=1}^{N} \log\cosh(\beta\langle\xi_i, z\rangle) \tag{54}$$

$$\langle\xi_i, z\rangle := \sum_{\mu=1}^{M} \xi_i^\mu z^\mu. \tag{55}$$

The limit behavior of the measure $\tilde{\mathcal{Q}}_{N,\xi,\beta}$ can then be studied by using Laplace's method: we first investigate the minima of $\Phi_{N,\xi,\beta}$ and then show that indeed these minima determine the asymptotic behavior of $\tilde{\mathcal{Q}}_{N,\xi,\beta}$. Since $(\xi_i)_{i=1\ldots N}$ are IID variables, one can apply a law of large numbers to show that $\Phi_{N,\xi,\beta}$ converges almost surely to:

$$\Phi_{\xi,\beta}(z) = \frac{\beta\|z\|^2}{2} - E^\xi \log\cosh(\beta\langle\xi_i, z\rangle) \tag{56}$$

Here the expectation is taken with respect to ξ_i. Using the estimate given in (Bovier and Gayrard (1997a,b), Theorem 4.1.), the limit behavior of $\tilde{\mathcal{Q}}_{N,\xi,\beta}$ is obtained by replacing $\Phi_{N,\xi,\beta}$ by its limit $\Phi_{\xi,\beta}$. The critical points of $\Phi_{\xi,\beta}$ can then be found by taking the derivative of (56) with respect to z:

$$\forall\alpha = 1\ldots M, \quad z^\alpha = E[\xi_i^\alpha \tanh(\beta < \xi_i, z >)] \tag{57}$$

In Lemma A.1 (see Appendix A) it is shown that the interesting solutions of this equation can only have a single non-zero coordinate: they are of the form $\pm z^\alpha(\beta)e^\alpha$ where $(e^\alpha, \alpha = 1\ldots M)$ is the canonical basis of R^M. It is then easy to see by substituting in Eq. (57) that $z^\alpha(\beta)$ is a solution of

$$z^\alpha = E[\xi_i^\alpha \tanh(\beta\xi_i^\alpha z^\alpha)] \tag{58}$$

First remark that $z^\alpha = 0$ is always a solution of this equation. Moreover, for $\beta \le \beta_\alpha = (4\pi^\alpha(1-\pi^\alpha))^{-1}$, it is the only solution. However when $\beta > \beta_\alpha$ two new solutions $\pm z^\alpha(\beta) \ne 0$ appear. This leads to $2M$ critical points for $\Phi_{\xi,\beta}$. In order to see which one(s) are global minima, assume without loss of generality that $1/2 \le \pi^1 = \pi^2 = \ldots = \pi^k < \pi^{k+1} \le \ldots \le \pi^M$. Then $\beta_1 = \ldots = \beta_k \le \beta_{k+1} \le \beta_M$. Moreover for a given β, the value of the minimum $\Phi_{\xi,\beta}(z^\alpha)$

$$\Phi_{\xi,\beta}(z^\alpha) = \frac{\beta z^\alpha(\beta)^2}{2} - \pi^\alpha \ln\cosh(2\beta(1-\pi^\alpha)z^\alpha(\beta)) - (1-\pi^\alpha)\ln\cosh(2\beta\pi^\alpha z^\alpha(\beta)) \tag{59}$$

increases with π^α: $\Phi_{\xi,\beta}(z^1(\beta)) \le \Phi_{\xi,\beta}(z^2(\beta)) \le \ldots \Phi_{\xi,\beta}(z^M(\beta))$ (see Fig. 3). The "deepest" minima are therefore located at $\pm z(\beta)e^1$. If $k > 1$ then there are $2k$ global minima situated on the set Λ defined in (27). Using Laplace's method (as in Gentz and Löwe, 1999), one can then show that for any bounded continuous function $f : R^M \to R$, the behavior of the integral

$$\int d\tilde{\mathcal{Q}}_{N,\xi,\beta}(z) f(z) \tag{60}$$

for large N is determined by the values of f on the finite set Λ. In particular if f vanishes on $\Lambda_{\epsilon/2}$ then the integral vanishes as $N \to \infty$. Given now any Borel set $A \subset R^M \setminus \Lambda_\epsilon$, since the indicator function 1_A can be approximated by bounded continuous functions vanishing on $\Lambda_{\epsilon/2}$ (32) follows. Note that due to the exponential form of the density in (52) even more is true: if $A \subset R^M \setminus \Lambda_\epsilon$ is measurable, then $\tilde{\mathcal{Q}}_{N,\xi,\beta}(A) \le e^{-N\delta}$ and also

$$\forall\epsilon > 0, \exists\delta > 0, \mathcal{Q}_{N,\xi,\beta}(A) \le e^{-N\delta} \quad \text{a.s. in}\,\xi.$$

(2) Random communication structure:

Define $H^0_{N,\xi}$ as the counterpart of $H_{N,\xi}$ when $p = 1$:

$$H^0_{N,\xi}(\Phi) := -\frac{1}{2N}\sum_{i,j=1}^{N}\sum_{\alpha=1}^{M} \xi_i^\alpha \xi_j^\alpha \phi_i \phi_j \tag{61}$$

$H^0_{N,\xi}$ determines the properties of the equilibrium distribution in the case $p = 1$ while $H_{N,\xi}$ determines the equilibrium in the general case $p < 1$ (random communication structure). Analogously, we define $\mathcal{Q}^0_{N,\xi,\beta}$ to be the distribution of $m_{N,\xi}$ when $p = 1$. To extend the convergence result above to the case $p < 1$, we use the following result from (Bovier and Gayrard, 1993) (see also Comets, 2002) which shows that, if $pN \to \infty$ then the difference between $H^0_{N,\xi}$ and $H_{N,\xi}$ can be controlled with probability arbitrarily close to one:

Lemma B.1 (Bovier and Gayrard, 1993, Proposition 3.1). *When $pN \to \infty$ there exists a set $\mathcal{C}_N \subset \Omega$ such that*

$$\mathbb{P}[\mathcal{C}_N] \geq \left(1 - \frac{K}{N^2}\right)(1 - e^{-\rho N}) \tag{62}$$

for some constants $K > 0$, $\rho > 0$ such that:

$$\forall \omega \in \mathcal{C}_N, \forall \phi \in \{-1, +1\}^N, \qquad |H_{N,\xi}(\phi)(\omega) - H^0_{N,\xi}(\phi)(\omega)| \leq \gamma(N)N. \tag{63}$$

where $\gamma(N) \downarrow 0$ as $N \uparrow \infty$ and $c > 0$ is such that $pN\gamma(N)^2 > c$.

To see why this helps recall that in the first part of this proof (where the communication structure is complete) we have proved the following: If a A fulfills $A \cap \Lambda^\epsilon_M = \varnothing$ for some $\epsilon > 0$ then not only do we have $\mathcal{Q}^0_{N,\xi,\beta}(A) \to 0$ but also

$$\mathcal{Q}^0_{N,\xi,\beta}(A) \leq e^{-N\delta}$$

for a $\delta > 0$ depending on A and M but chosen uniformly with respect to ξ on a set with probability $\geq 1 - K'/N^2$. This yields

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{Q}^0_{N,\xi,\beta}(A) \leq -\delta$$

Lemma B.1 above gives us a tool to compare $\mathcal{Q}_{N,\xi,\beta}$ to $\mathcal{Q}^0_{N,\xi,\beta}$: For all $\omega \in \mathcal{C}_N$ and all measurable sets $A \subset \mathbb{R}^M$ one has

$$e^{-2\gamma(N)N} \mathcal{Q}^0_{N,\xi,\beta}(A) \leq \mathcal{Q}_{N,\xi,\beta}(\omega, A) \leq e^{2\gamma(N)N} \mathcal{Q}^0_{N,\xi,\beta}(A). \tag{64}$$

While this estimate looks rather poor, it does enable us to control $\mathcal{Q}_{N,\xi,\beta}(A)$ for events A fulfilling $A \cap \Lambda^\epsilon_M = \varnothing$ for some $\epsilon > 0$. Indeed it implies that also

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{Q}_{N,\xi,\beta}(A) \leq -\delta$$

for such sets A. This in turn implies

$$\mathcal{Q}_{N,\xi,\beta}(A) \leq e^{-N\delta/2}$$

for N large enough. This yields the assertion of the theorem. □

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