Quantitative Finance, Vol. 10, No. 6, June–July 2010, 593–606

Robustness and sensitivity analysis of risk measurement procedures

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(*Received 18 December 2008; in final form 1 February 2010*)

Measuring the risk of a financial portfolio involves two steps: estimating the loss distribution of the portfolio from available observations and computing a 'risk measure' that summarizes the risk of the portfolio. We define the notion of 'risk measurement procedure', which includes both of these steps, and introduce a rigorous framework for studying the robustness of risk measurement procedures and their sensitivity to changes in the data set. Our results point to a conflict between the subadditivity and robustness of risk measurement procedures and show that the same risk measure may exhibit quite different sensitivities depending on the estimation procedure used. Our results illustrate, in particular, that using recently proposed risk measures such as CVaR/expected shortfall leads to a less robust risk measurement procedure than historical Value-at-Risk. We also propose alternative risk measurement procedures that possess the robustness property.

Keywords: Risk management; Risk measurement; Coherent risk measures; Law invariant risk measures; Value-at-Risk; Expected shortfall

1. Introduction

One of the main purposes of quantitative modeling in finance is to quantify the risk of financial portfolios. In connection with the widespread use of Value-at-Risk and related risk measurement methodologies and the Basel committee guidelines for risk-based requirements for regulatory capital, methodologies for measuring of the risk of financial portfolios have been the focus of recent attention and have generated a considerable theoretical literature (Artzner *et al.* 1999, Acerbi 2002, 2007, Föllmer and Schied 2002, 2004, Frittelli and Rosazza Gianin 2002). In this theoretical approach to risk measurement, a risk measure is represented as a map assigning a number (a measure of risk) to each random payoff. The focus of this literature has been on the properties of such maps and requirements for the risk measurement procedure to be coherent, in a static or dynamic setting.

Since most risk measures such as Value-at-Risk or Expected Shortfall are defined as functionals of the portfolio loss distribution, an implicit starting point is the knowledge of the loss distribution. In applications, however, this probability distribution is unknown and should be estimated from (historical) data as part of the risk measurement procedure. Thus, in practice, measuring the risk of a financial portfolio involves two steps: estimating the loss distribution of the portfolio from available observations and computing a risk measure that summarizes the risk of this loss distribution. While these two steps have been considered and studied separately, they are intertwined in applications and an important criterion in the choice of a risk measure is the availability of a method for accurately estimating it. Estimation or mis-specification errors in the portfolio loss distribution can have a considerable impact on risk measures, and it is important to examine the sensitivity of risk measures to these errors (Gourieroux *et al.* 2000, Gourieroux and Liu 2006).

1.1. A motivating example

Consider the following example, based on a data set of 1000 loss scenarios for a derivatives portfolio

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Quantitative Finance
ISSN 1469–7688 print/ISSN 1469–7696 online © 2010 Taylor & Francis
http://www.informaworld.com
DOI: 10.1080/14697681003685597

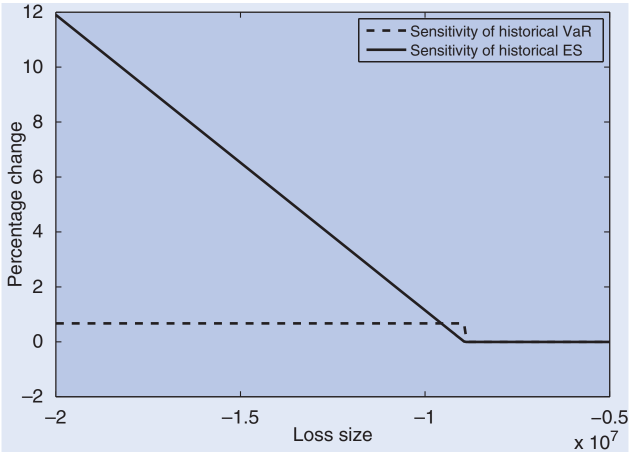


Figure 1. Empirical sensitivity (in percentage) of the historical VaR at 99% and historical ES at 99%.

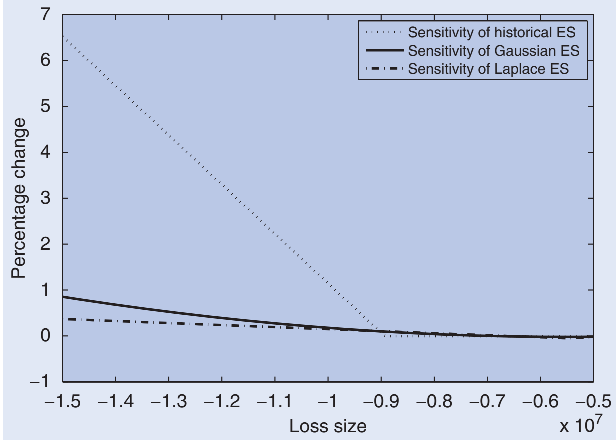


Figure 2. Empirical sensitivity (in percentage) of the ES at 99% estimated with diverse methods.

incorporating hundreds of different risk factors.† The historical Value-at-Risk (VaR) i.e. the quantile of the empirical loss distribution, and the Expected Shortfall (Acerbi 2002) of the empirical loss distribution, computed at the 99% level are, respectively, 8.887 M$ and 9.291 M$.

To examine the sensitivity of these estimators to the addition of a *single* observation in the data set, we compute the (relative) change (in %) in the estimators when a new observation is added to the data set. Figure 1 displays this measure of sensitivity as a function of the size of the additional observation. While the levels of the two risk measures are not very different, they display quite different sensitivities to a change in the data set, the Expected Shortfall being much more sensitive to large observations while VaR has a bounded sensitivity. While Expected Shortfall has the advantage of being a coherent risk measure (Artzner *et al.* 1999, Acerbi 2002), it appears to lack robustness with respect to small changes in the data set.

Another point, which has been left out of most studies on risk measures (with the notable exception of Gourieroux and Liu 2006), is the impact of the estimation method on these sensitivity properties. A risk measure such as Expected Shortfall (ES) can be estimated in different ways: either directly from the empirical loss distribution ('historical ES') or by first estimating a parametric model (Gaussian, Laplace, etc.) from the observed sample and computing the Expected Shortfall using the estimated distribution.

Figure 2 shows the sensitivity of the Expected Shortfall for the same portfolio as above, but estimated using three different methods. We observe that different estimators for the same risk measure exhibit very different sensitivities to an additional observation (or outlier).

These examples motivate the need for assessing the sensitivity and robustness properties of risk measures in conjunction with the estimation method being used to compute them. In order to study the interplay of a risk measure and its estimation method used for computing it, we define the notion of *risk measurement procedure* as a two-step procedure that associates with a payoff X and a data set $\mathbf{x} = (x_1, \ldots, x_n)$ of size n a risk estimate $\widehat{\rho}(\mathbf{x})$ based on the data set $\mathbf{x}$. This estimator of the 'theoretical' risk measure $\rho(X)$ is said to be robust if small variations in the loss distribution—resulting either from estimation or mis-specification errors—result in small variations in the estimator.

1.2. Contribution of the present work

In the present work, we propose a rigorous approach for examining how estimation issues can affect the computation of risk measures, with a particular focus on robustness and sensitivity analysis of risk measurement procedures, using tools from robust statistics (Huber 1981, Hampel *et al.* 1986). In contrast to the considerable literature on risk measures (Artzner *et al.* 1999, Kusuoka 2001, Acerbi 2002, 2007, Föllmer and Schied 2002, Rockafellar and Uryasev 2002, Tasche 2002), which does not discuss estimation issues, we argue that the choice of the estimation method and the risk measure should be considered jointly using the notion of *risk estimator*.

We introduce a qualitative notion of 'robustness' for a risk measurement procedure and a way of quantifying it via sensitivity functions. Using these tools we show that there is a conflict between coherence (more precisely, the subadditivity) of a risk measure and the robustness, in the statistical sense, of its commonly used estimators. This consideration goes against the traditional arguments for the use of coherent risk measures and therefore merits discussion. We complement this abstract result by computing measures of sensitivity, which allow us to quantify the robustness of various risk measures with respect to the data set used to compute them. In particular, we show that the same 'risk measure' may exhibit quite different sensitivities depending on the

†Data courtesy of Société Générale Risk Management unit.

estimation procedure used. These properties are studied in detail for some well-known examples of risk measures: Value-at-Risk, Expected Shortfall/CVaR (Acerbi 2002, Rockafellar and Uryasev 2002, Tasche 2002) and the class of spectral risk measures introduced by Acerbi (2007). Our results illustrate, in particular, that historical Value-at-Risk, while failing to be sub-additive, leads to a more robust procedure than alternatives such as Expected shortfall.

Statistical estimation and sensitivity analysis of risk measures have also been studied by Gourieroux *et al.* (2000) and Gourieroux and Liu (2006). In particular, Gourieroux and Liu (2006) consider non-parametric estimators of distortion risk measures (which includes the class studied in this paper) and focus on the asymptotic distribution of these estimators. By contrast, we study their robustness and sensitivity using tools from robust statistics.

Methods of robust statistics are known to be relevant in quantitative finance. Czellar *et al.* (2007) discuss robust methods for estimation of interest-rate models. Dell'Aquila and Embrechts (2006) discuss robust estimation in the context of extreme value theory. Heyde *et al.* (2007), which appeared simultaneously with the first version of this paper, discuss some ideas similar to those discussed here, but in a finite data set (i.e. non-asymptotic) framework. We show that, using appropriate definitions of consistency and robustness, the discussion can be extended to a large-sample/asymptotic framework which is the usual setting for discussion of estimators. Our asymptotic framework allows us to establish a clear link, absent in Heyde *et al.* (2007), between properties of risk estimators and those of risk measures.

1.3. Outline

Section 2 recalls some basic notions on distribution-based risk measures and establishes the distinction between a risk measure and a *risk measurement procedure*. We show that a risk measurement procedure applied to a data set can be viewed as the application of an *effective* risk measure to the empirical distribution obtained from this data and give examples of effective risk measures associated with various risk estimators.

Section 3 defines the notion of *robustness* for a risk measurement procedure and examines whether this property holds for commonly used risk measurement procedures. We show, in particular, that there exists a conflict between the subadditivity of a risk measure and the robustness of its estimation procedure.

In section 4 we define the notion of *sensitivity function* for a risk measure and compute sensitivity functions for some commonly used risk measurement procedures. In particular, we show that, while historical VaR has a bounded sensitivity to a change in the underlying data set, the sensitivity of Expected Shortfall estimators is unbounded. We discuss in section 5 some implications of our findings for the design of risk measurement procedures in finance.

2. Estimation of risk measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space representing market outcomes and L^0 be the space of all random variables. We denote by $\mathcal{D}$ the (convex) set of cumulative distribution functions (cdf) on $\mathbb{R}$. The distribution of a random variable X is denoted $F_X \in \mathcal{D}$, and we write $X \sim F$ if $F_X = F$. The Lévy distance (Huber 1981) between two cdfs $F, G \in \mathcal{D}$ is

$$d(F,G) \triangleq \inf\{\varepsilon > 0 : F(x-\varepsilon) - \varepsilon \leq G(x) \leq F(x+\varepsilon) + \varepsilon,\ \forall x \in \mathbb{R}\}.$$

The upper and lower quantiles of $F \in \mathcal{D}$ of order $\alpha \in (0,1)$ are defined, respectively, by

$$q_\alpha^+(F) \triangleq \inf\{x \in \mathbb{R} : F(x) > \alpha\} \geq q_\alpha^-(F) \triangleq \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}.$$

Abusing notation, we denote $q_\alpha^\pm(X) = q_\alpha^\pm(F_X)$. For $p \geq 1$ we denote by $\mathcal{D}^p$ the set of distributions having a finite pth moment, i.e.

$$\int_{\mathbb{R}} |x|^p \mathrm{d}F(x) < \infty,$$

and by $\mathcal{D}_-^p$ the set of distributions whose left tail has a finite p moment. We denote $\mu(F)$ the mean of $F \in \mathcal{D}^1$ and $\sigma^2(F)$ the variance of $F \in \mathcal{D}^2$. For any $n \geq 1$ and any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$F_{\mathbf{x}}^{\mathrm{emp}}(x) \triangleq \frac{1}{n}\sum_{i=1}^{n} I_{\{x \geq x_i\}}$$

denotes the empirical distribution of the data set $\mathbf{x}$; $\mathcal{D}_{\mathrm{emp}}$ will denote the set of all empirical distributions.

2.1. Risk measures

The 'Profit and Loss' (P&L) or payoff of a *portfolio* over a specified horizon may be represented as a random variable $X \in L \subset L^0(\Omega, \mathcal{F}, \mathbb{P})$, where negative values for X correspond to losses. The set L of such payoffs is assumed to be a convex cone containing all constants. A *risk measure* on L is a map $\rho : L \to \mathbb{R}$ assigning to each P&L $X \in L$ a number representing its degree of riskiness.

Artzner *et al.* (1999) advocated the use of *coherent* risk measures, defined as follows.

Definition 2.1 (coherent risk measure (Artzner *et al.* 1999)): A risk measure $\rho : L \to \mathbb{R}$ is *coherent* if it is

(1) *monotone* (*decreasing*): $\rho(X) \leq \rho(Y)$ provided $X \geq Y$;
(2) *cash-additive* (*additive with respect to cash reserves*): $\rho(X+c) = \rho(X) - c$ for any $c \in \mathbb{R}$;
(3) *positive homogeneous*: $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda \geq 0$;
(4) *sub-additive*: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

The vast majority of risk measures used in finance are statistical, or distribution-based risk measures, i.e. they depend on X only through its distribution F_X:

$$F_X = F_Y \Rightarrow \rho(X) = \rho(Y).$$

In this case, ρ can be represented as a map on the set of probability distributions, which we still denote by ρ. Therefore, by setting

$$\rho(F_X) \triangleq \rho(X),$$

we can view ρ as a map defined on (a subset of) the set of probability distributions $\mathcal{D}$.

We focus on the following class of distribution-based risk measures, introduced by Acerbi (2002) and Kusuoka (2001), which contains all examples used in the literature:

$$\rho_m(X) = -\int_0^1 q_u^-(X) m(\mathrm{d}u), \tag{1}$$

where m is a probability measure on (0, 1). Let $\mathcal{D}_m$ be the set of distributions of r.v. for which the above risk measure is finite. ρ_m can then be viewed as a map $\rho_m : \mathcal{D}_m \mapsto \mathbb{R}$. Notice that if the support of m does not contain 0 or 1, then $\mathcal{D}_m = \mathcal{D}$.

Three cases deserve particular attention.

- *Value at Risk (VaR)*. This is the risk measure that is frequently used in practice and corresponds to the choice $m = \delta_\alpha$ for a fixed $\alpha \in (0, 1)$ (usually $\alpha \le 10\%$), that is

  $$\mathrm{VaR}_\alpha(F) \triangleq -q_\alpha^-(F). \tag{2}$$

  Its domain of definition is all $\mathcal{D}$.
- *Expected shortfall (ES)*. This corresponds to choosing m as the uniform distribution over $(0, \alpha)$, where $\alpha \in (0, 1)$ is fixed:

  $$\mathrm{ES}_\alpha(F) \triangleq \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(F)\mathrm{d}u. \tag{3}$$

  In this case, $\mathcal{D}_m = \mathcal{D}_-^1$, the set of distributions having an integrable left tail.
- *Spectral risk measures* (Acerbi 2002, 2007). This class of risk measures generalizes ES and corresponds to choosing $m(\mathrm{d}u) = \phi(u)\mathrm{d}u$, where $\phi : [0, 1] \to [0, +\infty)$ is a density on [0, 1] and $u \mapsto \phi(u)$ is decreasing. Therefore,

  $$\rho_\phi(F) \triangleq \int_0^1 \mathrm{VaR}_u(F)\phi(u)\mathrm{d}u. \tag{4}$$

  If $\phi \in L^q(0, 1)$ (but not in $L^{q+\varepsilon}$) and $\phi \equiv 0$ around 1, then $\mathcal{D}_-^p \subset \mathcal{D}_m$, where $p^{-1} + q^{-1} = 1$.

For any choice of the weight m, ρ_m defined in (1) is monotone, additive with respect to cash and positive homogeneous. The subadditivity of such risk measures has been characterized as follows (Kusuoka 2001, Föllmer and Schied 2002, Acerbi 2007).

Proposition 2.2 (Kusuoka 2001, Föllmer and Schied 2002, Acerbi 2007)**:** *The risk measure ρ_m defined in* (1) *is sub-additive (hence coherent) on $\mathcal{D}_m$ if and only if it is a spectral risk measure.*

As a consequence, we recover the well-known facts that ES is a coherent risk measure, while VaR is not.

2.2. *Estimation of risk measures*

Once a (distribution-based) risk measure ρ has been chosen, in practice one has first to estimate the P&L distribution of the portfolio from available data and then apply the risk measure ρ to this distribution. This can be viewed as a two-step procedure.

(1) Estimation of the loss distribution F_X: one can use either an empirical distribution obtained from a historical or simulated sample or a parametric form whose parameters are estimated from available data. This step can be formalized as a function from $\mathcal{X} = \cup_{n\ge1}\mathbb{R}^n$, the collection of all possible data sets, to $\mathcal{D}$; if $\mathbf{x} \in \mathcal{X}$ is a data set, we denote $\widehat{F}_\mathbf{x}$ the corresponding estimate of F_X.

(2) Application of the risk measure ρ to the estimated P&L distribution $\widehat{F}_\mathbf{x}$, which yields an estimator $\widehat{\rho}(\mathbf{x}) \triangleq \rho(\widehat{F}_\mathbf{x})$ for $\rho(X)$.

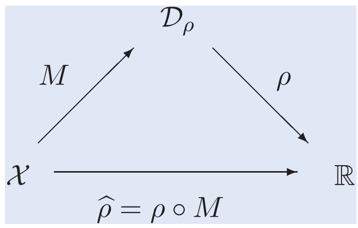
We call the combination of these two steps a *risk measurement procedure*.

Definition 2.3 (risk measurement procedure)**:** A *risk measurement procedure* (RMP) is a couple (M, ρ), where $\rho : \mathcal{D}_\rho \to \mathbb{R}$ is a risk measure and $M : \mathcal{X} \to \mathcal{D}_\rho$ an estimator for the loss distribution.

The outcome of this procedure is a *risk estimator* $\widehat{\rho} : \mathcal{X} \to \mathbb{R}$ defined as

$$\mathbf{x} \mapsto \widehat{\rho}(\mathbf{x}) \triangleq \rho(\widehat{F}_\mathbf{x}),$$

that estimates $\rho(X)$ given the data $\mathbf{x}$ (see diagram).



2.2.1. Historical risk estimators.

The *historical estimator* $\widehat{\rho}^{\mathrm{h}}$ associated with a risk measure ρ is the estimator obtained by applying ρ to the empirical P&L distribution (sample cdf) $\widehat{F}_\mathbf{x} = F_\mathbf{x}^{\mathrm{emp}}$:

$$\widehat{\rho}^{\mathrm{h}}(\mathbf{x}) = \rho(F_\mathbf{x}^{\mathrm{emp}}).$$

For a risk measure ρ_m, as in (1),

$$\widehat{\rho}_m^{\mathrm{h}}(\mathbf{x}) = \rho_m(F_\mathbf{x}^{\mathrm{emp}}) = -\sum_{i=1}^n w_{n,i}x_{(i)}, \quad \mathbf{x} \in \mathbb{R}^n,$$

where $x_{(k)}$ is the kth least element of the set $\{x_i\}_{i\le n}$, and the weights are equal to

$$w_{n,i} \triangleq m\left(\frac{i-1}{n}, \frac{i}{n}\right] \quad \text{for } i = 1, \ldots, n-1,$$
$$w_{n,n} = m\left(\frac{n-1}{n}, 1\right).$$

Historical estimators are L-estimators in the sense of Huber (1981).

Example 2.4: Historical VaR_α is given by

$$\widehat{\mathrm{VaR}}^{\mathrm{h}}_\alpha(\mathbf{x}) = -x_{(\lfloor n\alpha \rfloor + 1)}, \tag{5}$$

where $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{R}$.

Example 2.5: The historical expected shortfall ES_α is given by

$$\widehat{\mathrm{ES}}^{\mathrm{h}}_\alpha(\mathbf{x}) = -\frac{1}{n\alpha}\left(\sum_{i=1}^{\lfloor n\alpha \rfloor} x_{(i)} + x_{(\lfloor n\alpha \rfloor + 1)}(n\alpha - \lfloor n\alpha \rfloor)\right). \tag{6}$$

Example 2.6: The historical estimator of the spectral risk measure ρ_ϕ associated with ϕ is given by

$$\widehat{\rho}^{\mathrm{h}}_\phi(\mathbf{x}) = -\sum_{i=1}^{n} w_{n,i} x_{(i)}, \quad \text{where } w_{n,i} = \int_{(i-1)/n}^{i/n} \phi(u)\mathrm{d}u. \tag{7}$$

2.2.2. Maximum likelihood estimators. In the parametric approach to loss distribution modeling, a parametric model is assumed for F_X and parameters are estimated from data using, for instance, maximum likelihood. We call the risk estimator obtained the 'maximum likelihood risk estimator' (MLRE). We discuss these estimators for scale families of distributions, which include as a special case (although a multidimensional one) the common variance–covariance method for VaR estimation. Let F be a centered distribution. The scale family generated by the reference distribution F is defined by

$$\mathcal{D}_F \triangleq \{F(\cdot \mid \sigma) : \sigma > 0\}, \quad \text{where } F(x \mid \sigma) \triangleq F\left(\frac{x}{\sigma}\right).$$

If $F \in \mathcal{D}^p$ $(p \geq 1)$, then $\mathcal{D}_F \subset \mathcal{D}^p$ and it is common to choose F with location 0 and scale 1, so that $F(\cdot|\sigma)$ has location 0 and scale σ^2. In line with common practice in short-term risk management we assume that the risk factor changes have location equal to zero. Two examples of scale families of distributions that we will study are:

- the Gaussian family where F has density
$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right),$$
- the Laplace or double exponential family where F has density
$$f(x) = \frac{1}{2}\exp(-|x|).$$

The Maximum Likelihood Estimator (MLE) $\widehat{\sigma} = \widehat{\sigma}^{\mathrm{mle}}(\mathbf{x})$ of σ is defined by

$$\widehat{\sigma} = \arg.\max_{\sigma>0} \sum_{i=1}^{n} \ln f(x_i \mid \sigma), \tag{8}$$

and solves the following nonlinear equation:

$$\sum_{i=1}^{n} x_i \frac{f'(x_i/\widehat{\sigma})}{f(x_i/\widehat{\sigma})} = -n\widehat{\sigma}. \tag{9}$$

Let ρ be a positively homogeneous risk measure, then we have

$$\rho(F(\cdot \mid \sigma)) = \rho(F)\sigma.$$

Therefore, if the scale parameter is estimated by maximum likelihood, the associated risk estimator of ρ is then given by

$$\widehat{\rho}(\mathbf{x}) = c\widehat{\sigma}^{\mathrm{mle}}(\mathbf{x}).$$

Example 2.7 (MLRE for a Gaussian family)**:** The MLE of the scale parameter in the Gaussian scale family is

$$\widehat{\sigma}(\mathbf{x}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}. \tag{10}$$

The resulting risk estimators are given by $\widehat{\rho}(\mathbf{x}) = c\widehat{\sigma}(\mathbf{x})$, where, depending on the risk measure considered, c is given by

$$c = \mathrm{VaR}_\alpha(F) = -z_\alpha,$$
$$c = \mathrm{ES}_\alpha(F) = \frac{\exp\{-z_\alpha^2/2\}}{\alpha\sqrt{2\pi}},$$
$$c = \rho_\phi(F) = -\int_0^1 z_u \phi(u)\mathrm{d}u,$$

where z_α is the α-quantile of a standard normal distribution.

Example 2.8 (ML risk estimators for Laplace distributions)**:** The MLE of the scale parameter in the Laplace scale family is

$$\widehat{\lambda}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} |x_i|. \tag{11}$$

Note that this scale estimator is *not* the standard deviation but the Mean Absolute Deviation (MAD). The resulting risk estimator is

$$\widehat{\rho}(\mathbf{x}) = c\widehat{\lambda}(\mathbf{x}), \tag{12}$$

where c takes the following values, depending on the risk measure considered (we assume $\alpha \leq 0.5$):

$$c = \mathrm{VaR}_\alpha(F) = -\ln(2\alpha),$$
$$c = \mathrm{ES}_\alpha(F) = 1 - \ln(2\alpha),$$
$$c = \rho_\phi(F) = -\int_0^{1/2} \ln(2u)\phi(u)\mathrm{d}u + \int_{1/2}^{1} \ln(2-2u)\phi(u)\mathrm{d}u.$$

2.3. Effective risk measures

In all of the above examples we observe that the risk estimator $\widehat{\rho}(\mathbf{x})$, computed from a data set $\mathbf{x} = (x_1, \ldots, x_n)$, can be expressed in terms of the empirical distribution $F^{\mathrm{emp}}_{\mathbf{x}}$; in other words, there exists a risk measure ρ_{eff} such that, for any data set $\mathbf{x} = (x_1, \ldots, x_n)$, the risk estimator $\widehat{\rho}(\mathbf{x})$ is equal to the new risk measure ρ_{eff} applied to the empirical distribution

$$\rho_{\mathrm{eff}}(F^{\mathrm{emp}}_{\mathbf{x}}) \triangleq \widehat{\rho}(\mathbf{x}). \tag{13}$$

We will call ρ_{eff} the *effective risk measure* associated with the risk estimator $\widehat{\rho}$. In other words, while ρ is the risk measure we are interested in computing, the effective risk measure ρ_{eff} is the risk measure that the procedure defined in definition 2.3 *actually computes*.

So far, the effective risk measure ρ_{eff} is defined for all empirical distributions by (13). Consider now a risk estimator $\widehat{\rho}$ that is *consistent* with the risk measure ρ at $F \in \mathcal{D}_\rho$, that is

$$\widehat{\rho}(X_1, \ldots, X_n) \overset{n\to\infty}{\to} \rho(F) \quad \text{a.s.},$$

for any i.i.d. sequence $X_i \sim F$. Consistency of a risk estimator for a class of distributions of interest is a minimal requirement to ask for. If $\widehat{\rho}$ is consistent with the risk measure ρ for $F \in \mathcal{D}_{\text{eff}} \subset \mathcal{D}_\rho$, we can extend ρ_{eff} to $\mathcal{D}_{\text{eff}}$ as follows: for any sequence $(x_i)_{i\geq 1}$ such that

$$\frac{1}{n}\sum_{i=1}^{n} I_{\{\cdot \geq x_i\}} \overset{d}{\to} F(\cdot) \in \mathcal{D}_{\text{eff}},$$

we define

$$\rho_{\text{eff}}(F) \triangleq \lim_{n\to\infty} \widehat{\rho}(x_1, \ldots, x_n). \tag{14}$$

Consistency guarantees that $\rho_{\text{eff}}(F)$ is independent of the chosen sequence.

Definition 2.9 (effective risk measure)**:** Let $\widehat{\rho} : \mathcal{X} \to \mathbb{R}$ be a consistent risk estimator of a risk measure ρ for a class $\mathcal{D}_{\text{eff}}$ of distributions. There is a unique risk measure $\rho_{\text{eff}} : \mathcal{D}_{\text{eff}} \mapsto \mathbb{R}$ such that

- $\rho_{\text{eff}}(F_{\mathbf{x}}^{\text{emp}}) \triangleq \widehat{\rho}(\mathbf{x})$ for any data set $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}$;
- $\rho_{\text{eff}}(F) \triangleq \lim_{n\to\infty} \widehat{\rho}(x_1, \ldots, x_n)$ for any sequence $(x_i)_{i\geq 1}$ such that

$$\frac{1}{n}\sum_{i=1}^{n} I_{\{\cdot \geq x_i\}} \overset{d}{\to} F(\cdot) \in \mathcal{D}_{\text{eff}}.$$

Equation (13), defining the effective risk measure, allows us in most examples to characterize ρ_{eff} explicitly. As shown in the examples below, ρ_{eff} may be quite different from ρ and lacks many of the properties ρ was initially chosen for.

Example 2.10 (historical VaR)**:** The empirical quantile $\widehat{\text{VaR}}_\alpha^{\text{h}}$ is a consistent estimator of VaR_α for any $F \in \mathcal{D}$ such that $q_\alpha^+(F) = q_\alpha^-(F)$. Otherwise, $\widehat{\text{VaR}}_\alpha^{\text{h}}(X_1, \ldots, X_n)$ may not have a limit as $n \to \infty$. Therefore, the effective risk measure associated with $\widehat{\text{VaR}}_\alpha^{\text{h}}$ is VaR_α restricted to the set

$$\mathcal{D}_{\text{eff}} = \{F \in \mathcal{D} : q_\alpha^+(F) = q_\alpha^-(F)\}.$$

Example 2.11 (historical estimator of ES and spectral risk measures)**:** A general result on L-estimators by Van Zwet (1980) implies that the historical estimator of any spectral risk measure ρ_ϕ (in particular of the ES) is consistent with ρ_ϕ at any F where the risk measure is defined. Therefore, the effective risk measure associated with $\widehat{\rho}_\phi^{\text{h}}$ coincides with ρ_ϕ. The same remains true even if the density ϕ is not decreasing, so that ρ_ϕ is not a spectral risk measure.

Example 2.12 (Gaussian ML risk estimator)**:** Consider the risk estimator introduced in example 2.7. The associated effective risk measure is defined on $\mathcal{D}^2$ and given by

$$\rho_{\text{eff}}(F) = c\sigma(F), \quad \text{where } \sigma(F) = \sqrt{\int_{\mathbb{R}} x^2 \, \mathrm{d}F(x)}.$$

Example 2.13 (Laplace ML risk estimator)**:** Consider the risk estimators introduced in example 2.8. The associated effective risk measure is defined on $\mathcal{D}^1$ and given by

$$\rho_{\text{eff}}(F) = c\lambda(F), \quad \text{where } \lambda(F) = \int_{\mathbb{R}} |x| \mathrm{d}F(x).$$

Notice that, in both of these examples, the effective risk measure ρ_{eff} is different from the original risk measure ρ.

3. Qualitative robustness

We now define the notion of *qualitative robustness* of a risk estimator and use it to examine the robustness of the various risk estimators considered above.

3.1. $\mathcal{C}$-Robustness of a risk estimator

Fix a set $\mathcal{C} \subseteq \mathcal{D}$ representing the set of 'plausible' loss distributions, containing all the empirical distributions: $\mathcal{D}_{\text{emp}} \subseteq \mathcal{C}$. In most examples the class $\mathcal{C}$ is specified via an integrability condition (e.g., existence of moments) in which case it automatically contains all empirical distributions. Consider an interior element $F \in \mathcal{C}$, i.e. for any $\delta > 0$, there exists $G \in \mathcal{C}$, with $G \neq F$, such that $d(G, F) \leq \delta$. We call a (risk) estimator $\mathcal{C}$-robust at F if the law of the estimator is continuous with respect to a change in F (remaining in $\mathcal{C}$) *uniformly* in the size n of the data set.

Definition 3.1: A risk estimator $\widehat{\rho}$ is $\mathcal{C}$-*robust* at F if, for any $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \geq 1$ such that, for all $G \in \mathcal{C}$,

$$d(G, F) \leq \delta \Longrightarrow d(\mathcal{L}_n(\widehat{\rho}, G)), \mathcal{L}_n(\widehat{\rho}, F)) \leq \varepsilon, \quad \forall n \geq n_0,$$

where d is the Lévy distance.

When $\mathcal{C} = \mathcal{D}$, i.e. when any perturbation of the distribution is allowed, the previous definition corresponds to the notion of *qualitative robustness* (also called asymptotic robustness) as outlined by Huber (1981). This case is not generally interesting in econometric or financial applications since requiring robustness against all perturbations of the model F is quite restrictive and excludes even estimators such as the sample mean.

Obviously, the larger the set of perturbations $\mathcal{C}$, the harder it is for a risk estimator to be $\mathcal{C}$-robust. In the remainder of this section we will assess whether the risk estimators previously introduced are $\mathcal{C}$-robust w.r.t. a suitable set of perturbations $\mathcal{C}$.

3.2. Qualitative robustness of historical risk estimators

The following generalization of a result of Hampel *et al.* (1986) is crucial for the analysis of robustness of historical risk estimators.

Proposition 3.2: *Let ρ be a risk measure and $F \in \mathcal{C} \subseteq \mathcal{D}_\rho$. If $\widehat{\rho}^h$, the historical estimator of ρ, is consistent with ρ at every $G \in \mathcal{C}$, then the following are equivalent:*

(1) *the restriction of ρ to $\mathcal{C}$ is continuous (w.r.t. the Lévy distance) at F;*
(2) *$\widehat{\rho}^h$ is $\mathcal{C}$-robust at F.*

A proof of proposition 3.2 is given in appendix A. From this proposition, we obtain the following corollary that provides a sufficient condition on the risk measure to ensure that the corresponding historical/empirical estimator is robust.

Corollary 3.3: *If ρ is continuous in $\mathcal{C}$, then $\widehat{\rho}^h$ is $\mathcal{C}$-robust at any $F \in \mathcal{C}$.*

Proof: Fix $G \in \mathcal{C}$ and let $(X_n)_{n \geq 1}$ be an i.i.d. sequence distributed as G. Then, by the Glivenko–Cantelli Theorem we have, for almost all ω,

$$d(F^{\text{emp}}_{\mathbf{X}(\omega)}, G) \xrightarrow{n\to\infty} 0, \quad \mathbf{X} = (X_1, \ldots, X_n).$$

By continuity of ρ at G it holds that, again for almost all ω,

$$\widehat{\rho}(\mathbf{X}(\omega)) = \rho(F^{\text{emp}}_{\mathbf{X}(\omega)}) \to \rho(G),$$

and therefore $\widehat{\rho}$ is consistent with ρ at G. A simple application of proposition 3.2 concludes. □

Our analysis will use the following important result adapted from Huber (1981, theorem 3.1). For a measure m on $[0, 1]$ let

$$\mathcal{A}_m \triangleq \{\alpha \in [0,1] : m(\{\alpha\}) > 0\}$$

be the set of values where m puts a positive mass. We remark that $\mathcal{A}_m$ is empty for a continuous m (as in the definition of spectral risk measures).

Theorem 3.4: *Let ρ_m be a risk measure of the form* (1). *If the support of m does not contain* 0 *or* 1, *then ρ_m is continuous at any $F \in \mathcal{D}_\rho$ such that $q^+_\alpha(F) = q^-_\alpha(F)$ for any $\alpha \in \mathcal{A}_m$. Otherwise, ρ_m is not continuous at any $F \in \mathcal{D}_\rho$.*

In other words, a risk measure of the form (1) can be continuous at some F if and only if its computation does not involve any extreme quantile (close to 0 or 1.) In this case, continuity is ensured provided F is continuous at all points where m has a point mass.

3.2.1. Historical VaR$_\alpha$.

In this case, $\mathcal{A}_m = \{\alpha\}$, so combining corollary 3.3 and theorem 3.4 we have the following.

Proposition 3.5: *The historical estimator of VaR_α is $\mathcal{C}_\alpha$-robust at any $F \in \mathcal{C}_\alpha$, where*

$$\mathcal{C}_\alpha \triangleq \{F \in \mathcal{D} : q^+_\alpha(F) = q^-_\alpha(F)\}.$$

In other words, if the quantile of the (true) loss distribution is uniquely determined, then the empirical quantile is a robust estimator.

3.2.2. Historical estimator of ES and spectral risk measures.

Let ρ_ϕ defined in (4) be in terms of a density ϕ in $L^q(0, 1)$, so that $\mathcal{D}_\rho = \mathcal{D}^p$ (p and q are conjugate.) However, here we do *not* assume that ϕ is decreasing, so that ρ_ϕ need not be a spectral risk measure, although it is still in the form (1).

Proposition 3.6: *For any $F \in \mathcal{D}^p$, the historical estimator of ρ_ϕ is $\mathcal{D}^p$-robust at F if and only if, for some $\varepsilon > 0$,*

$$\phi(u) = 0, \quad \forall u \in (0, \varepsilon) \cup (1 - \varepsilon, 1), \tag{15}$$

i.e. ϕ vanishes in the neighborhood of 0 *and* 1.

Proof: We have seen in section 2.3 that $\widehat{\rho}^{\text{h}}_\phi$ is consistent with ρ_ϕ at any $F \in \mathcal{D}_\rho$. If (15) holds for some ε, then the support of m (recall that $m(\mathrm{d}u) = \phi(u)\mathrm{d}u$) does not contain 0 or 1. As $\mathcal{A}_m$ is empty, theorem 3.4 yields continuity of ρ at any distribution in $\mathcal{D}_\rho$. Hence, we have $\mathcal{D}_\rho$-robustness of $\widehat{\rho}$ at F thanks to corollary 3.3.

On the contrary, if (15) does not hold for any ε, then 0 or 1 (or both) are in the support of m and therefore ρ_ϕ is not continuous at any distribution in $\mathcal{D}_\rho$, in particular at F. Therefore, by proposition 3.2 we conclude that $\widehat{\rho}$ is not $\mathcal{D}_\rho$-robust at F. □

An immediate, but important consequence is the following.

Corollary 3.7: *The historical estimator of any spectral risk measure ρ_ϕ defined on $\mathcal{D}^p$ is* not *$\mathcal{D}^p$-robust at any $F \in \mathcal{D}^p$. In particular, the historical estimator of ES_α is not $\mathcal{D}^1$-robust at any $F \in \mathcal{D}^1$.*

Proof: It is sufficient to observe that, for a spectral risk measure, the density ϕ is decreasing and therefore it cannot vanish around 0, otherwise it would vanish on the entire interval [0, 1]. □

Proposition 3.6 illustrates a conflict between subadditivity and robustness: as soon as we require a (distribution-based) risk measure ρ_m to be coherent, its historical estimator fails to be robust (at least when all possible perturbations are considered).

3.2.3. A robust family of risk estimators.

We have just seen that ES$_\alpha$ has a non-robust historical estimator. However, we can remove this drawback by slightly modifying its definition. Consider $0 < \alpha_1 < \alpha_2 < 1$ and define the risk measure

$$\frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \mathrm{VaR}_u(F)\mathrm{d}u.$$

This is simply the average of VaR levels across a range of loss probabilities. As

$$\phi(u) = \frac{1}{\alpha_2 - \alpha_1} I_{\alpha_1 < u < \alpha_2}$$

vanishes around 0 and 1, proposition 3.6 shows that the historical (i.e. using the empirical distribution directly) risk estimator of this risk measure is $\mathcal{D}^1$-robust. Of course, the corresponding risk measure is not coherent since ϕ is not decreasing. Note that, for $\alpha_1 < 1/n$, where n is the sample size, this risk estimator is indistinguishable from the historical Expected Shortfall! Yet, unlike Expected Shortfall estimators, it has good robustness properties as $n \to \infty$. One can also consider a discrete version of the above risk measure:

$$\frac{1}{k}\sum_{j=1}^{k} \mathrm{VaR}_{u_j}(F), \quad 0 < u_1 < \cdots < u_k < 1,$$

which enjoys similar robustness properties.

3.3. *Qualitative robustness of the maximum likelihood risk estimator*

We now discuss the qualitative robustness for MLRE in a scale family of a risk measure ρ_m defined as in (1). First, we generalize the definition of the scale Maximum Likelihood Estimator $\widehat{\sigma}^{\mathrm{mle}}(\mathbf{x}) \triangleq \sigma^{\mathrm{mle}}(F_{\mathbf{x}}^{\mathrm{emp}})$ given in equations (8) and (9) for distributions that do not belong to $\mathcal{D}_{\mathrm{emp}}$.

Let $\mathcal{D}_F$ be the scale family associated with the distribution $F \in \mathcal{D}$ and assume that $\mu(F)=0$, $\sigma(F)=1$ and F admits a density f. Define the function

$$\psi(x) \triangleq \left\{\frac{\partial}{\partial\sigma}\ln\left[\frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)\right]\right\}_{\sigma=1} = -1 - x\frac{f'(x)}{f(x)}, \quad x \in \mathbb{R}. \tag{16}$$

The ML estimator $\sigma^{\mathrm{mle}}(G)$ of the scale parameter $\sigma(G)$ for $G \in \mathcal{D}_F$ corresponds to the unique σ that solves

$$\gamma(\sigma, G) \triangleq \int \psi\left(\frac{x}{\sigma}\right) G(\mathrm{d}x) = 0, \quad \text{for } G \in \mathcal{D}_F. \tag{17}$$

By defining $\mathcal{D}_\psi = \{G \in \mathcal{D} : \int |\psi(x)| G(\mathrm{d}x) < \infty\}$, we can extend the definition of $\sigma^{\mathrm{mle}}(G)$ to all $G \in \mathcal{D}_\psi$. Note that when $G \notin \mathcal{D}_F$, $\sigma^{\mathrm{mle}}(G)$ may exist if $G \in \mathcal{D}_\psi$ but does not correspond to the ML estimator of the scale parameter of G. Moreover, from definition (17), we notice that if we compute the ML estimator of the scale parameter for a distribution $G_{\mathbf{x}}^{\mathrm{emp}} \in \mathcal{D}_{\mathrm{emp}}$ we recover the MLE $\widehat{\sigma}^{\mathrm{mle}}(\mathbf{x})$ introduced in equations (8) and (9). In the examples below, we have computed the function ψ for the Gaussian and Laplace scale families.

Example 3.8 (Gaussian scale family)**:** The function ψ^{g} for the Gaussian scale family is

$$\psi^{\mathrm{g}}(x) = -1 + x^2, \tag{18}$$

and we immediately conclude that $\mathcal{D}_{\psi^{\mathrm{g}}} = \mathcal{D}^2$.

Example 3.9 (Laplace scale family)**:** The function ψ^{l} for the Laplace scale family is equal to

$$\psi^{\mathrm{l}}(x) = -1 + |x|, \tag{19}$$

and we obtain $\mathcal{D}_{\psi^{\mathrm{l}}} = \mathcal{D}^1$.

The following result exhibits conditions on the function ψ under which the MLE of the scale parameter is weakly continuous on $\mathcal{D}_\psi$.

Theorem 3.10 (weak continuity of the scale MLE)**:** *Let $\mathcal{D}_F$ be the scale family associated with the distribution $F \in \mathcal{D}$ and assume that $\mu(F)=0$, $\sigma(F)=1$ and F admits a density f. Suppose now that ψ, defined as in* (16), *is even, increasing on $\mathbb{R}^+$, and takes values of both signs. Then, the following two assertions are equivalent*:

- *$\sigma^{mle} : \mathcal{D}_\psi \mapsto \mathbb{R}^+$, defined as in* (17), *is weakly continuous at $F \in \mathcal{D}_\psi$*;
- *ψ is bounded and $\gamma(\sigma, F) \triangleq \int \psi(x/\sigma)F(\mathrm{d}x) = 0$ has a unique solution $\sigma = \sigma^{mle}(F)$ for all $F \in \mathcal{D}_\psi$.*

A proof is given in appendix A. Using the above result, we can now study the qualitative robustness of parametric risk estimators for Gaussian or Laplace scale families.

Proposition 3.11 (non-robustness of Gaussian and Laplace MLRE)**:** *Gaussian* (*respectively Laplace*) *MLRE of cash-additive and homogeneous risk measures are not $\mathcal{D}^2$-robust* (*respectively $\mathcal{D}^1$-robust*) *at any F in $\mathcal{D}^2$* (*respectively in $\mathcal{D}^1$*).

Proof: We detail the proof for the Gaussian scale family. The same arguments hold for the Laplace scale family. Let us consider a Gaussian MLRE of a translation invariant and homogeneous risk measure, denoted by $\widehat{\rho}(\mathbf{x}) = c\,\widehat{\sigma}^{\mathrm{mle}}(\mathbf{x})$. First of all we notice that the function ψ^{g} associated with the MLE of the scale parameter of a distribution belonging to the Gaussian scale family is even, and increasing on $\mathbb{R}^+$. Moreover, it takes values of both signs. Secondly, we recall that the effective risk measure associated with the Gaussian ML risk estimator is $\rho_{\mathrm{eff}}(F) = c\sigma^{\mathrm{mle}}(F)$ for all $F \in \mathcal{D}_{\mathrm{eff}} = \mathcal{D}_{\psi^{\mathrm{g}}} = \mathcal{D}^2$. Therefore, as ψ^{g} is unbounded, by using theorem 3.10, we know that ρ_{eff} is not continuous at any $F \in \mathcal{D}^2$. As the Gaussian MLRE considered $\widehat{\rho}$ verifies $\widehat{\rho}(\mathbf{x}) = \widehat{\rho_{\mathrm{eff}}}^{\mathrm{h}}(\mathbf{x})$, and is consistent with ρ_{eff} at all $F \in \mathcal{D}^2$ by construction, we can apply proposition 3.2 to conclude that, for $F \in \mathcal{D}^2$, $\widehat{\rho}$ is not $\mathcal{D}^2$-robust at F. □

4. Sensitivity analysis

In order to quantify the degree of robustness of a risk estimator, we now introduce the concept of the *sensitivity function*.

Definition 4.1 (sensitivity function of a risk estimator)**:** The *sensitivity function* of a risk estimator at $F \in \mathcal{D}_{\mathrm{eff}}$ is the function defined by

$$S(z) = S(z; F) \triangleq \lim_{\varepsilon \to 0^+} \frac{\rho_{\mathrm{eff}}(\varepsilon\delta_z + (1-\varepsilon)F) - \rho_{\mathrm{eff}}(F)}{\varepsilon},$$

for any $z \in \mathbb{R}$ such that the limit exists.

$S(z; F)$ measures the sensitivity of the risk estimator to the addition of a new data point in a large sample. It corresponds to the directional derivative of the effective risk measure ρ_{eff} at F in the direction $\delta_z \in \mathcal{D}$. In the language of robust statistics, $S(z;F)$ is the influence function (Costa and Deshayes 1977, Deniau *et al.* 1977, Huber 1981, Hampel *et al.* 1986) of the effective risk measure ρ_{eff} and is related to the asymptotic variance of the historical estimator of ρ (Huber 1981, Hampel *et al.* 1986).

Remark 1: If $\mathcal{D}_\rho$ is convex and contains all empirical distributions, then $\varepsilon\delta_z+(1-\varepsilon)F\in\mathcal{D}_\rho$ for any $\varepsilon\in[0,1]$, $z\in\mathbb{R}$ and $F\in\mathcal{D}_\rho$. These conditions hold for all the risk measures we are considering.

4.1. Historical VaR

We have seen above that the effective risk measure associated with $\widehat{\text{VaR}}^{\text{h}}_\alpha$ is the restriction of VaR_α to

$$\mathcal{D}_{\text{eff}}=\mathcal{C}_\alpha=\{F\in\mathcal{D}:q^+_\alpha(F)=q^-_\alpha(F)\}.$$

The sensitivity function of the historical VaR_α has the following simple explicit form.

Proposition 4.2: *If $F\in\mathcal{D}$ admits a strictly positive density f, then the sensitivity function at F of the historical VaR_α is*

$$S(z)=\begin{cases}\frac{1-\alpha}{f(q_\alpha(F))}, & \text{if } z<q_\alpha(F),\\ 0, & \text{if } z=q_\alpha(F),\\ -\frac{\alpha}{f(q_\alpha(F))}, & \text{if } z>q_\alpha(F).\end{cases}\tag{20}$$

Proof: First we observe that the map $u\mapsto q(u)\triangleq q_u(F)$ is the inverse of F and so it is differentiable at any $u\in(0,1)$ and we have

$$q'(u)=\frac{1}{F'(q(u))}=\frac{1}{f(q_u(F))}.$$

Fix $z\in\mathbb{R}$ and set, for $\varepsilon\in[0,1)$, $F_\varepsilon=\varepsilon\delta_z+(1-\varepsilon)F$, so that $F\equiv F_0$. For $\varepsilon>0$, the distribution F_ε is differentiable at any $x\neq z$, with $F'_\varepsilon(x)=(1-\varepsilon)f(x)>0$, and has a jump (of size ε) at the point $x=z$. Hence, for any $u\in(0,1)$ and $\varepsilon\in[0,1)$, $F_\varepsilon\in\mathcal{C}_u$, i.e. $q^-_u(F_\varepsilon)=q^+_u(F_\varepsilon)\triangleq q_u(F_\varepsilon)$. In particular,

$$q_\alpha(F_\varepsilon)=\begin{cases}q\left(\frac{\alpha}{1-\varepsilon}\right), & \text{for } \alpha<(1-\varepsilon)F(z),\\ q\left(\frac{\alpha-\varepsilon}{1-\varepsilon}\right), & \text{for } \alpha\geq(1-\varepsilon)F(z)+\varepsilon,\\ z, & \text{otherwise.}\end{cases}\tag{21}$$

Assume now that $z>q(\alpha)$, i.e. $F(z)>\alpha$; from (21) it follows that

$$q_\alpha(F_\varepsilon)=q\left(\frac{\alpha}{1-\varepsilon}\right),\quad \text{for } \varepsilon<1-\frac{\alpha}{F(z)}.$$

As a consequence,

$$\begin{aligned}S(z)&=\lim_{\varepsilon\to0^+}\frac{\text{VaR}_\alpha(F_\varepsilon)-\text{VaR}_\alpha(F_0)}{\varepsilon}=-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}q_\alpha(F_\varepsilon)|_{\varepsilon=0}\\&=-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}q\left(\frac{\alpha}{1-\varepsilon}\right)\Big|_{\varepsilon=0}=-\left[\frac{1}{f(q(\alpha/(1-\varepsilon)))}\frac{\alpha}{(1-\varepsilon)^2}\right]_{\varepsilon=0}\\&=-\frac{\alpha}{f(q_\alpha(F))}.\end{aligned}$$

The case $q(\alpha)<z$ is handled in a very similar way. Finally, if $z=q(\alpha)$, then, again by (21), we have $q_\alpha(F_\varepsilon)=z$ for any $\varepsilon\in[0,1)$. Hence,

$$S(z)=-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}q_\alpha(F_\varepsilon)|_{\varepsilon=0}=0,$$

and we conclude. □

This example shows that the historical VaR_α has a bounded sensitivity to a change in the data set, which means that this risk estimator is not very sensitive to a small change in the data set.

4.2. Historical estimators of Expected Shortfall and spectral risk measures

We now consider a spectral risk measure ρ_ϕ defined by a weight function ϕ as in (4).

Proposition 4.3: *Consider a distribution F with a density $f>0$. Assume the following.*

(1)
$$\int_0^1\frac{u}{f(q_u(F))}\phi(u)\mathrm{d}u<\infty.$$

(2) *The risk measure is only sensitive to the left tail (losses)*
$$\exists u_0\in(0,1),\quad \forall u>u_0,\quad \phi(u)=0.$$

(3) *The density f is increasing in the left tail: $\exists x_0<0$, f increasing on $(-\infty, x_0)$.*

The sensitivity function at $F\in\mathcal{D}_\phi$ of the historical estimator of ρ_ϕ is then given by

$$S(z)=-\int_0^{F(z)}\frac{u}{f(q_u(F))}\phi(u)\mathrm{d}u+\int_{F(z)}^1\frac{1-u}{f(q_u(F))}\phi(u)\mathrm{d}u.$$

We note that the assumptions in the proposition are verified for Expected Shortfall and for all commonly used parametric distributions in finance: Gaussian, Student-*t*, double-exponential, Pareto (with exponent >1), etc.

Proof: Using the notation introduced in the proof of proposition 4.2 we have

$$S(z)=\lim_{\varepsilon\to0^+}\int_0^1\frac{\text{VaR}_u(F_\varepsilon)-\text{VaR}_u(F)}{\varepsilon}\phi(u)\mathrm{d}u.$$

We will now show that the order of the integral and the limit $\epsilon\to0$ can be interchanged using a dominated convergence argument. First note that $\lim_{\varepsilon\to0^+}\varepsilon^{-1}(\text{VaR}_u(F_\varepsilon))-\text{VaR}_u(F))$ exists, and is finite for all $u\in(0,1)$. Define $\bar{u}=\inf\{F(x_0),\,u_0\}$ and consider $\epsilon_0>0$. By the mean value theorem, for any $u\leq\bar{u}(1-\varepsilon)$:

$$\forall\epsilon\leq\epsilon_0,\quad \exists\xi\in(0,\epsilon),$$
$$\text{VaR}_u(F_\varepsilon)=\text{VaR}_u(F_0)-\varepsilon\frac{u}{(1-\xi)^2f(q(u/(1-\xi)))}.$$

Therefore, we have

$$\left|\frac{\mathrm{VaR}_u(F_\varepsilon)-\mathrm{VaR}_u(F)}{\varepsilon}\right| \le \frac{u}{(1-\xi)^2 f(q(u/(1-\xi)))} \le \frac{u}{(1-\varepsilon_0)^2 f(q(u/(1-\xi)))}, \quad \xi<\varepsilon\le\varepsilon_0, \le \frac{u}{(1-\varepsilon_0)^2 f(q(u))} \in L^1(\phi), \quad f, q \text{ increasing},$$

so that we can apply dominated convergence

$$S(z)=\int_0^1 \lim_{\varepsilon\to 0^+}\frac{\mathrm{VaR}_u(F_\varepsilon)-\mathrm{VaR}_u(F)}{\varepsilon}\phi(u)\mathrm{d}u = \int_0^1 -\left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}q_u(F_\varepsilon)\right]_{\varepsilon=0}\phi(u)\mathrm{d}u = -\int_0^{F(z)}\frac{u}{f(q_u(F))}\phi(u)\mathrm{d}u+\int_{F(z)}^1\frac{1-u}{f(q_u(F))}\phi(u)\mathrm{d}u,$$

thanks to proposition 4.2. □

Since the effective risk measure associated with historical ES_α is ES_α itself, defined on $\mathcal{D}^1_- = \{F\in\mathcal{D} : \int x^- F(\mathrm{d}x)<\infty\}$, an immediate consequence of the previous proposition is the following.

Corollary 4.4: *The sensitivity function at $F\in\mathcal{D}^1_-$ for historical ES_α is*

$$S(z)=\begin{cases} -\frac{z}{\alpha}+\frac{1-\alpha}{\alpha}q_\alpha(F)-ES_\alpha(F), & \text{if } z\le q_\alpha(F),\\ -q_\alpha(F)-ES_\alpha(F), & \text{if } z\ge q_\alpha(F).\end{cases}$$

This result shows that the sensitivity of historical ES_α is linear in z, and thus unbounded. This means that this risk measurement procedure is less robust than the historical VaR_α.

4.3. ML risk estimators for Gaussian distributions

We have seen that the effective risk measure associated with Gaussian maximum likelihood estimators of VaR, ES, or any spectral risk measure is

$$\rho_{\mathrm{eff}}(F)=c\sigma(F), \quad F\in\mathcal{D}_{\mathrm{eff}}=\mathcal{D}^2,$$

where $c=\rho(Z)$, $Z\sim N(0,1)$, is a constant depending only on the risk measure ρ (we are not interested in its explicit value here).

Proposition 4.5: *The sensitivity function at $F\in\mathcal{D}^2$ of the Gaussian ML risk estimator of a positively homogeneous risk measure ρ is*

$$S(z)=\frac{\sigma c}{2}\left[\left(\frac{z}{\sigma}\right)^2-1\right].$$

Proof: Let, for simplicity, $\sigma=\sigma(F)$. Fix $z\in\mathbb{R}$ and set, as usual, $F_\varepsilon=(1-\varepsilon)F+\varepsilon\delta_z$ ($\varepsilon\in[0,1)$); observe that $F_\varepsilon\in\mathcal{D}^2$ for any ε. If we set $\upsilon(\varepsilon)\triangleq c\sigma(F_\varepsilon)$, with $c=\rho(N(0,1))$, then we have $S(z)=\upsilon'(0)$. It is immediate to compute

$$\sigma^2(F_\varepsilon)=\int_{\mathbb{R}}x^2F_\varepsilon(\mathrm{d}x)=(1-\varepsilon)\int_{\mathbb{R}}x^2F(\mathrm{d}x)+\varepsilon z^2-\varepsilon z^2 = (1-\varepsilon)(\sigma^2)+\varepsilon z^2-\varepsilon^2z^2=\sigma^2+\varepsilon[z^2-\sigma^2]-\varepsilon^2\sigma^2.$$

As a consequence,

$$\upsilon'(0)=c\left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\sqrt{\sigma^2(F_\varepsilon)}\right]_{\varepsilon=0}=\frac{\sigma c}{2}\left[\left(\frac{z}{\sigma}\right)^2-1\right].$$

□

4.4. ML risk estimators for Laplace distributions

We have seen that the effective risk measure of the Laplace MLRE of VaR, ES, or any spectral risk measure is

$$\rho_{\mathrm{eff}}(F)=c\lambda(F), \quad F\in\mathcal{D}^1,$$

where $c=\rho(G)$, G is the distribution with density $g(x)=\mathrm{e}^{-|x|}/2$, and $\lambda(F)=\int_{\mathbb{R}}|x|\,\mathrm{d}F(x)$.

Proposition 4.6: *Let ρ be a positively homogeneous risk measure. The sensitivity function at $F\in\mathcal{D}^1$ of its Laplace MLRE is*

$$S(z)=c(|z|-\lambda(F)).$$

Proof: As usual, we have, for $z\in\mathbb{R}$, $S(z)=\upsilon'(0)$, where $\upsilon(\varepsilon)=c\lambda(F_\varepsilon)$, $F_\varepsilon=(1-\varepsilon)F+\varepsilon\delta_z$ and c is defined above. We have

$$\upsilon(\varepsilon)=c(1-\varepsilon)\lambda(F)+c\varepsilon|z|,$$

and we conclude that

$$\upsilon'(0)=c|z|-c\lambda(F).$$

□

This proposition shows that the sensitivity of the Laplace MLRE at any $F\in\mathcal{D}^1$ is not bounded, but linear in z. Nonetheless, the sensitivity of the Gaussian MLRE is quadratic at any $F\in\mathcal{D}^2$, which indicates a greater sensitivity to outliers in the data set.

Table 1. Behavior of sensitivity functions for some risk estimators.

Risk estimator	Dependence in z of $S(z)$
Historical VaR	Bounded
Gaussian ML for VaR	Quadratic
Laplace ML for VaR	Linear
Historical Expected Shortfall	Linear
Gaussian ML for Expected Shortfall	Quadratic
Laplace ML for Expected Shortfall	Linear

4.5. Finite-sample effects

The sensitivity functions computed above are valid for (asymptotically) large samples. In order to assess the finite-sample relevance accuracy of risk estimator sensitivities, we compare them with the finite-sample sensitivity

$$S_N(z;F)=\frac{\widehat{\rho}(X_1,\ldots,X_N,z)-\widehat{\rho}(X_1,\ldots,X_N)}{1/(N+1)}.$$

Figure 3 compares the empirical sensitivities of historical, Gaussian, and Laplace VaR and historical, Gaussian, and Laplace ES with their theoretical (large sample) counterparts. We have used the same set of $N=1000$ loss scenarios as in section 1.1.

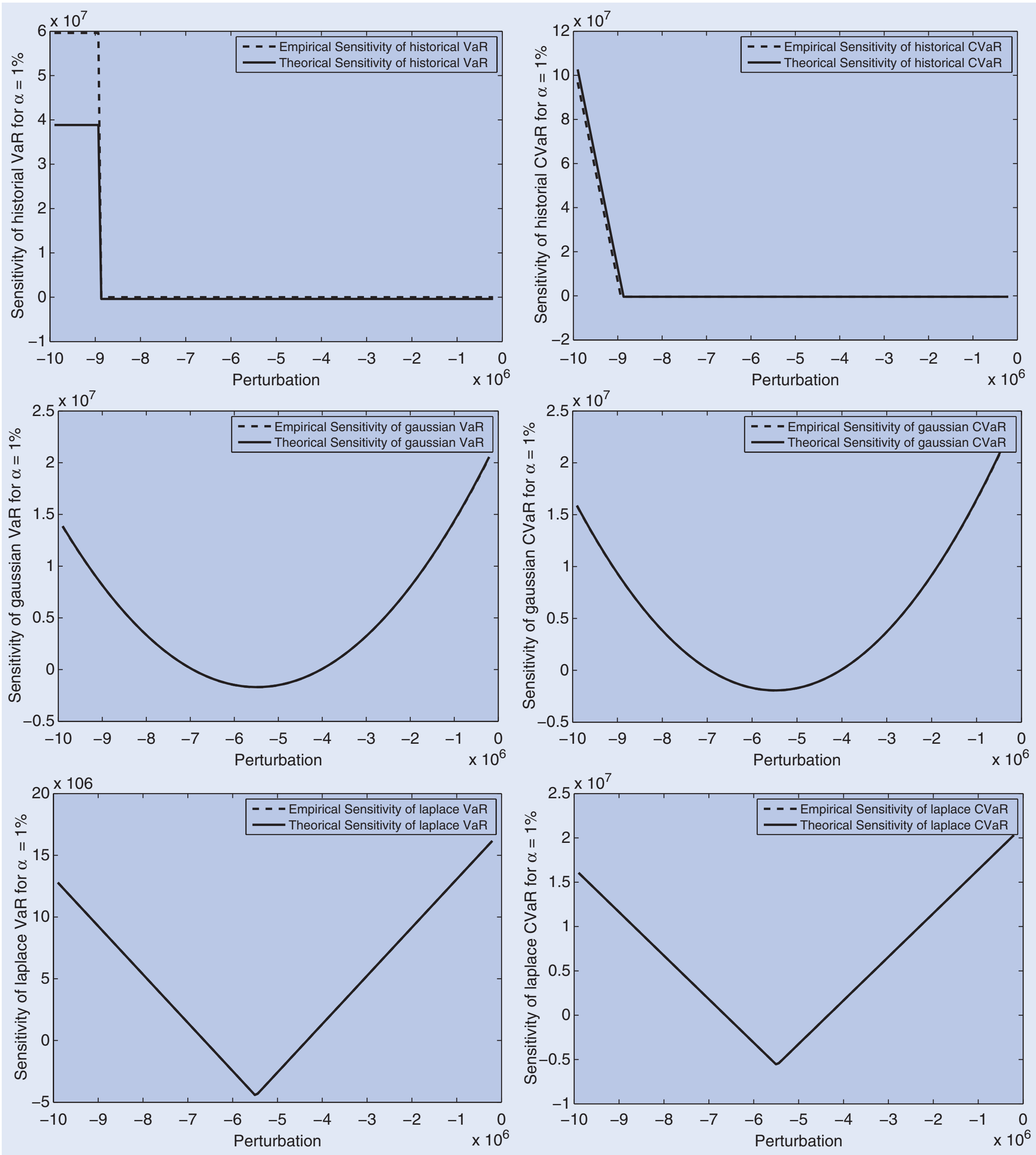


Figure 3. Empirical versus theoretical sensitivity functions of risk estimators for $\alpha = 1\%$ at a 1 day horizon. Historical VaR (upper left), historical ES (upper right), Gaussian VaR (left), Gaussian ES (right), Laplace VaR (lower left), Laplace ES (lower right).

The asymptotic and empirical sensitivities coincide for all risk estimators except for historical risk measurement procedures. For the historical ES, the theoretical sensitivity is very close to the empirical one. Nonetheless, we note that the empirical sensitivity of the historical VaR can be equal to 0 because it is strongly dependent on the integer part of $N\alpha$, where N is the number of scenarios and α the quantile level. This dependency disappears asymptotically for large samples.

The excellent agreement shown in these examples illustrates that the expressions derived above for theoretical sensitivity functions are useful for evaluating the sensitivity of risk estimators for realistic sample sizes. This is useful since theoretical sensitivity functions are analytically computable, whereas empirical sensitivities require perturbating the data sets and recomputing the risk measures.

5. Discussion

5.1. Summary of main results

Let us now summarize the contributions and main conclusions of this study.

First, we have argued that when the estimation step is explicitly taken into account in a risk measurement procedure, issues like robustness and sensitivity to the data set are important and need to be accounted for with at least the same attention as the coherence properties set forth by Artzner *et al.* (1999). Indeed, we do think that it is crucial for regulators and end-users to understand the robustness and sensitivity properties of the risk estimators they use or design to assess the capital requirement, or manage their portfolio. Indeed, an unstable/non-robust risk estimator, be it related to a coherent measure of risk, is of little use in practice.

Second, we have shown that the choice of the estimation method matters when discussing the robustness of risk measurement procedures: our examples show that different estimation methods coupled with the same risk measure lead to very different properties in terms of robustness and sensitivity.

Historical VaR is a qualitatively robust estimation procedure, whereas the proposed examples of coherent (distribution-based) risk measures do not pass the test of qualitative robustness and show high sensitivity to 'outliers'. This explains perhaps why many practitioners have been reluctant to adopt 'coherent' risk measures. Also, most parametric estimation procedures for VaR and ES lead to non-robust estimators. On the other hand, weighted averages of historical VaR such as

$$\frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \mathrm{VaR}_u(F)\mathrm{d}u,$$

with $1 > \alpha_2 > \alpha_1 > 0$, have robust empirical estimators.

5.2. Re-examining subadditivity

The conflict we have noted between robustness of a risk measurement procedure and the subadditivity of the risk measure shows that one cannot achieve robust estimation in this framework while preserving subadditivity. While a strict adherence to the coherence axioms of Artzner *et al.* (1999) would lead to choosing subadditivity over robustness, several recent studies (Danielsson *et al.* 2005, Heyde *et al.* 2007, Ibragimov and Walden 2007, Dhaene *et al.* 2008) have provided reasons for not doing so.

Danielsson *et al.* (2005) explore the potential for violations of VaR subadditivity and report that, for most practical applications, VaR is sub-additive. They conclude that, in practical situations, there is no reason to choose a more complicated risk measure than VaR, solely for reasons of subadditivity. Arguing in a different direction, Ibragimov and Walden (2007) show that, for very 'heavy-tailed' risks defined in a very general sense, diversification does not necessarily decrease tail risk but actually can increase it, in which case requiring subadditivity would in fact be unnatural. Finally, Finally, Heyde *et al.* (2007) argue against subadditivity from an *axiomatic* viewpoint and propose to replace it by a weaker property of *co-monotonic subadditivity*. All these objections to the subadditivity axiom deserve serious consideration and further support the choice of robust risk measurement procedures over non-robust procedures for the sole reason of saving subadditivity.

5.3. Beyond distribution-based risk measures

While the 'axiomatic' approach to risk measurement in principle embodies a much wider class of risk measures than distribution-based (or 'law-invariant') risk measures, research has almost exclusively focused on this rather restrictive class of risk measures. Our result, that coherence and robustness cannot coexist within this class, can also be seen as an argument for going beyond distribution-based risk measures. This also makes sense in the context of the ongoing discussion on systemic risk: evaluating exposure to systemic risk requires considering the joint distribution of a portfolio's losses with other, external, risk factors, not just the marginal distribution of its losses. In fact, risk measures that are not distribution-based are routinely used in practice: the Standard Portfolio Analysis of Risk (SPAN) margin system, and cited as the original motivation by Artzner *et al.* (1999), is a well-known example of such a method used by many clearinghouses and exchanges.

We hope to have convinced the reader that there is more to risk measurement than the choice of a 'risk measure': statistical robustness, and not only 'coherence', should be a concern for regulators and end-users when choosing or designing risk measurement procedures. The design of robust risk estimation procedures requires the explicit inclusion of the statistical estimation step in the analysis of the risk measurement procedure. We hope this work will stimulate further discussion on these important issues.

Acknowledgements

We thank Hans Föllmer, Peter Bank, Paul Embrechts, Gerhard Stahl and seminar participants at QMF 2006 (Sydney), Humboldt University (Berlin), Torino, Lecce, INFORMS Applied Probability Days (Eindhoven), the Cornell ORIE Seminar, the Harvard Statistics Seminar and EDHEC for helpful comments. This project has benefited from partial funding by the European Research Network 'Advanced Mathematical Methods for Finance' (AMAMEF).

References

Acerbi, C., Spectral measures of risk: a coherent representation of subjective risk aversion. *J. Bank. Finance*, 2002, **26**, 1505–1518.

Acerbi, C., Coherent measures of risk in everyday market practice. *Quant. Finance*, 2007, **7**, 359–364.

Artzner, P., Delbaen, F., Eber, J. and Heath, D., Coherent measures of risk. *Math. Finance*, 1999, **9**, 203–228.

Costa, V. and Deshayes, J., Comparaison des RLM estimateurs. Theorie de la robustesse et estimation d'un parametre. *Asterisque*, 1977, 43–44.

Czellar, V., Karolyi, G.A. and Ronchetti, E., Indirect robust estimation of the short-term interest rate process. *J. Empir. Finance*, 2007, **14**, 546–563.
Danielsson, J., Jorgensen, B., Samorodnitsky, G., Sarma, M. and de Vries, C., Subadditivity re-examined: the case for Value-at-Risk. Preprint, London School of Economics, 2005.
Dell'Aquila, R. and Embrechts, P., Extremes and robustness: a contradiction? *Financial Mkts Portfol. Mgmt*, 2006, **20**, 103–118.
Deniau, C., Oppenheim, G. and Viano, C., Courbe d'influence et sensibilite. Theorie de la robustesse et estimation d'un parametre. *Asterisque*, 1977, 43–44.
Dhaene, J., Laeven, R., Vanduffel, S., Darkiewicz, G. and Goovaerts, M., Can a coherent risk measure be too subadditive? *J. Risk Insurance*, 2008, **75**, 365–386.
Föllmer, H. and Schied, A., Convex measures of risk and trading constraints. *Finance Stochast*., 2002, **6**, 429–447.
Föllmer, H. and Schied, A., *Stochastic Finance: An Introduction in Discrete Time*, 2004 (Walter de Gruyter: Berlin).
Frittelli, M. and Rosazza Gianin, E., Putting order in risk measures. *J. Bank. Finance*, 2002, **26**, 1473–1486.
Gourieroux, C., Laurent, J. and Scaillet, O., Sensitivity analysis of values at risk. *J. Empir. Finance*, 2000, **7**, 225–245.
Gourieroux, C. and Liu, W., Sensitivity analysis of distortion risk measures. Working Paper, 2006.
Hampel, F., Ronchetti, E., Rousseeuw, P. and Stahel, W., *Robust Statistics: The Approach Based on Influence Functions*, 1986 (Wiley: New York).
Heyde, C., Kou, S. and Peng, X., What is a good risk measure: bridging the gaps between data, coherent risk measures, and insurance risk measures. Preprint, Columbia University, 2007.
Huber, P., *Robust Statistics*, 1981 (Wiley: New York).
Ibragimov, R. and Walden, J., The limits of diversification when losses may be large. *J. Bank. Finance*, 2007, **31**, 2551–2569.
Kusuoka, S., On law invariant coherent risk measures. *Adv. Math. Econ*., 2001, **3**, 83–95.
Rockafellar, R. and Uryasev, S., Conditional Value-at-Risk for general distributions. *J. Bank. Finance*, 2002, **26**, 1443–1471.
Tasche, D., Expected shortfall and beyond. In *Statistical Data Analysis Based on the L_1-Norm and Related Methods*, edited by Y. Dodge, pp. 109–123, 2002 (Birkhäuser: Boston, Basel, Berlin).
Van Zwet, W., A strong law for linear functions of order statistics. *Ann. Probab*., 1980, **8**, 986–990.

Appendix A: Proofs

A.1. Proof of proposition 3.2

'1. $\Rightarrow$ 2'. Assume that $\rho|_{\mathcal{C}}$ is continuous at F and fix $\varepsilon>0$. Since for all $G\in\mathcal{C}$

$$d(\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},F),\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G)) \le d(\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},F),\delta_{\rho(F)})+d(\delta_{\rho(F)},\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G)),$$

and $\widehat{\rho}^{\,\mathrm{h}}$ is consistent with ρ at F, it suffices to prove that there exists $\delta>0$ and $n_0\ge 1$ such that, for all $G\in\mathcal{C}$,

$$d(G,F)\le\delta \Longrightarrow d(\delta_{\rho(F)},\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G))\le\frac{\varepsilon}{2},\quad \forall n\ge n_0.$$

Note that Strassen's Theorem (Huber 1981, theorem 3.7) gives us the following sufficient condition to obtain the desired result:

$$\mathbb{P}\Big(|\rho(F)-\rho(G^{\mathrm{emp}}_{\mathbf{x}})|\le\frac{\varepsilon}{2}\Big)\ge 1-\frac{\varepsilon}{2}\Longrightarrow d(\delta_{\rho(F)},\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G))\le\frac{\varepsilon}{2}.$$

Then, by using that $\rho|_{\mathcal{C}}$ is continuous at F, there exists $\bar{\delta}>0$ such that, for each $G\in\mathcal{C}$ that satisfies $d(F,G)\le 2\bar{\delta}$, we have $d(\delta_{\rho(F)},\delta_{\rho(G)})=|\rho(F)-\rho(G)|<\varepsilon/2$. As $\mathcal{D}_{\mathrm{emp}}\subseteq\mathcal{C}$, we obtain that

$$\mathbb{P}(d(F,G^{\mathrm{emp}}_{\mathbf{x}})\le 2\bar{\delta})\le\mathbb{P}\Big(|\rho(F)-\rho(G^{\mathrm{emp}}_{\mathbf{x}})|\le\frac{\varepsilon}{2}\Big).$$

Therefore, it suffices to show that $\mathbb{P}(d(F,G^{\mathrm{emp}}_{\mathbf{x}})\le 2\bar{\delta})\ge 1-(\varepsilon/2)$.

Now, using that Glivenko–Cantelli convergence is uniform in G, we have for each $\varepsilon>0$ and $\delta>0$ the existence of $n_0\ge 1$ such that, for all $G\in\mathcal{C}$,

$$\mathbb{P}(d(G,G^{\mathrm{emp}}_{\mathbf{x}})\le\delta)\ge 1-\frac{\varepsilon}{2},\quad \forall n\ge n_0.$$

Therefore, the $\mathcal{C}$-robustness follows from the triangular inequality $d(F,G^{\mathrm{emp}}_{\mathbf{x}})\le d(F,G)+d(G,G^{\mathrm{emp}}_{\mathbf{x}})$ and by taking $\delta\le\bar{\delta}$.

'2. $\Rightarrow$ 1'. Conversely, assume that $\widehat{\rho}^{\,\mathrm{h}}$ is $\mathcal{C}$-robust at F and fix $\varepsilon>0$. Then there exists $\delta>0$ and $n_0\ge 1$ such that, for all $G\in\mathcal{C}$,

$$d(F,G)<\delta\Rightarrow d(\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},F),\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G))<\frac{\varepsilon}{3},\quad \forall n\ge n_0.$$

As a consequence, from the triangular inequality

$$\begin{aligned}|\rho(F)-\rho(G)| &= d(\delta_{\rho(F)},\delta_{\rho(G)})\\ &\le d(\delta_{\rho(F)},\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},F))+d(\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},F),\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G))\\ &\quad + d(\mathcal{L}_n(\widehat{\rho}^{\,\mathrm{h}},G),\delta_{\rho(G)}),\end{aligned}$$

and the consistence of $\widehat{\rho}^{\,\mathrm{h}}$ with ρ at F and any $G\in\mathcal{C}$, it follows that $\rho|_{\mathcal{C}}$ is continuous at F.

A.2. Proof of theorem 3.10

We will show that the continuity problem of the ML scale function $\sigma^{\mathrm{mle}}:\mathcal{D}_\psi\to\mathbb{R}^+$ of portfolios X can be reduced to the continuity (on a properly defined space) issue of the ML location function of portfolios $Y=\ln(X^2)$. The change of variable here is made to use the results of Huber (1981) concerning the weak continuity of location parameters. The distribution F can be seen as the distribution of a portfolio X_0 with $\mu(X_0)=0$ and $\sigma(X_0)=1$. Then, by setting $Y_0=\ln(X_0^2)$, and denoting by G the distribution of Y_0 we have

$$G(y)=P(Y_0\le y)=P(X_0^2\le \mathrm{e}^{y})=F(\mathrm{e}^{y/2})-F(-\mathrm{e}^{y/2}),$$
$$g(y)=G'(y)=\mathrm{e}^{y/2}f(\mathrm{e}^{y/2}).$$

Moreover, by introducing the function

$$\varphi(y)=-\frac{g'(y)}{g(y)},\quad \mathcal{D}_\varphi\triangleq\Big\{G:\int\varphi(y)G(\mathrm{d}y)<\infty\Big\},$$

we can define, as in Huber (1981), the ML location function $\mu^{\mathrm{mle}}(H)$ for any distribution $H\in\mathcal{D}_\varphi$ as the solution of the following implicit relation

$$\int\varphi(y-\mu)H(\mathrm{d}y)=0. \qquad \text{(A1)}$$

Now we consider the distribution $F_X\in\mathcal{D}_\psi$ of the random variable X representing the P&L of a portfolio and assume that F_X has density f_X and that the solution to $\int\psi(x/\sigma)F_X(\mathrm{d}x)=0$ has a unique solution $\sigma=\sigma^{\mathrm{mle}}(F_X)$. Denoting by F_Y the distribution of $Y=\ln(X^2)$, it is easy to check that $F_Y\in\mathcal{D}_\varphi$ since, for $y=\ln(x^2)$, we have

$$\begin{aligned}\varphi(y) &= -\frac{g'(y)}{g(y)}\\ &= -\frac{\frac{1}{2}\mathrm{e}^{(y)/2}f(\mathrm{e}^{(y)/2})+\frac{1}{2}\mathrm{e}^{y}f'(\mathrm{e}^{(y)/2})}{\mathrm{e}^{(y)/2}f(\mathrm{e}^{(y)/2})}\\ &= -\frac{1}{2}\left[1+\mathrm{e}^{(y)/2}\frac{f'(\mathrm{e}^{(y)/2})}{f(\mathrm{e}^{(y)/2})}\right]\\ &= -\frac{1}{2}\left[1+x\frac{f'(x)}{f(x)}\right]=\frac{1}{2}\psi(x).\end{aligned} \tag{A2}$$

Noticing that

$$F_Y(\mathrm{d}y)=f_Y(y)\mathrm{d}y=xf_X(x)\mathrm{d}(\ln(x^2))=2f_X(x)\mathrm{d}x=2F_X(\mathrm{d}x),$$

we immediately obtain from equations (17), (A1) and (A2) that $\sigma^{\mathrm{mle}}(F_X)=\mu^{\mathrm{mle}}(F_Y)$ when $Y=2\ \ln(X)$. We have therefore shown that a scale function characterized by the function ψ can also be interpreted as a location function characterized by the function φ. From equation (A2), we see that, for all $x\in\mathbb{R}$, $2\varphi(x)=\psi[\mathrm{e}^{x/2}]$. Therefore, as ψ is assumed to be even and increasing on $\mathbb{R}^+$, it implies that φ is increasing on $\mathbb{R}$. Moreover, as ψ takes values of both signs it is also true for φ. To conclude, we apply theorem 2.6 of Huber (1981), which states that a location function associated with φ is weakly continuous at G if and only if φ is bounded and the location function computed at G is unique.