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MODELING TERM STRUCTURE DYNAMICS: AN INFINITE DIMENSIONAL APPROACH

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Motivated by stylized statistical properties of interest rates, we propose a modeling approach in which the forward rate curve is described as a stochastic process in a space of curves. After decomposing the movements of the term structure into the variations of the short rate, the long rate and the deformation of the curve around its average shape, this deformation is described as the solution of a stochastic evolution equation in an infinite dimensional space of curves. In the case where deformations are local in maturity, this equation reduces to a stochastic PDE, of which we give the simplest example. We discuss the properties of the solutions and show that they capture in a parsimonious manner the essential features of yield curve dynamics: imperfect correlation between maturities, mean reversion of interest rates, the structure of principal components of forward rates and their variances. In particular we show that a flat, constant volatility structures already captures many of the observed properties. Finally, we discuss parameter estimation issues and show that the model parameters have a natural interpretation in terms of empirically observed quantities.

Keywords: Term structure of interest rates; forward rates; multifactor models; Hilbert space; stochastic PDE; random field.

Measuring the risk of fixed income portfolios requires a correct representation of the statistical properties of interest rate fluctuations and the ability to simulate realistic scenarios for future changes in the yield curve. This task is made difficult by the multivariate and correlated nature of interest rate movements: the state variable is not simply a set of numbers but a *curve* — the forward rate curve — with certain geometric and dynamic properties.

Our objective in this paper is to present a model of term structure movements which is both analytically tractable and reproduces the stylized empirical observations on interest rates with a small number of parameters. Our approach, inspired by the empirical observations of Bouchaud *et al.* [5, 6] (see also [3, 27]), is to represent the forward rate term structure as a randomly fluctuating curve subject to noise. Using Musiela's parametrization, we introduce a family of models in which the forward rate process is represented as a random field governed by a second order stochastic PDE. Our approach accounts for the fact that only a small number of factors seem to govern the covariance structure of term structure movements, without imposing *ad hoc* restrictions on the number of factors driving different rates or the form of the volatility functions. More importantly, the shapes and variances of principal components are obtained as a *result* of the model rather than an input.

The paper is structured as follows. Section 1 discusses the motivation for our model in relation with empirical observations on variations in interest rates. Based on these observations, we discuss in Sec. 2 the ingredients one should incorporate into an interest rate model in order to reproduce the observed statistical properties and propose give a mathematical formulation in terms of a stochastic evolution equation in a space of functions. In the case where only *local* deformations are allowed, this equation reduces to a stochastic partial differential equation: we study a simple example of such an evolution equation in Sec. 3 and show that, albeit its rudimentary structure, it reproduces many properties of term structure deformations in a simple manner. This formulation is compared with the alternative *hyperbolic* formulation: we show that a hyperbolic SPDE would lead to empirically undesirable features for yield curve dynamics.

1. Term Structure Models in the Light of Empirical Facts

1.1. *Risk-neutral vs statistical modeling*

There are various motivations for modeling term structure dynamics. The first is concerned with the pricing of interest rate derivative securities. In this context, which has been the principal motivation behind term structure models in the mathematical finance literature [17, 31], the main concern has been the development of "coherent" — in the sense of arbitrage-free — pricing criteria for securities whose payoffs depend on movements of interest rates.

The second motivation is the statistical description of the fluctuations of interest rates in view of risk measurement and management or optimization of trading strategies in the fixed income market. In contrast to the preceding approach where the emphasis is on cross-sectional coherence of prices given by the model, here the emphasis is on describing and reproducing as closely as possible the observed time evolution of interest rates from a statistical point of view. Such an approach is useful if one is interested in simulating evolution scenarios for yield curves, calculating Value-at-Risk of fixed-income positions but also from a theoretical point of view, to gain a better understanding of interest rate fluctuations and their relations to other economic variables.

From a mathematical point of view, the first approach corresponds to modeling the dynamics of interest rates under a *risk-neutral* (or forward-neutral) measure, in coherence with a cross section of observed option prices, while the second approach corresponds to the modeling of real-world dynamics of interest rates, based on historical observations.

Most of the existing work in the mathematical finance literature on interest rate modeling has emphasized the modeling of arbitrage-free yield curve dynamics under a "risk-neutral" measure. The reason for this trend is that, contrarily to a stock market, in a bond market the future values of many securities — zero coupon bonds — are known with certainty. The observability of the contemporaneous prices of these bonds then makes it possible to calibrate a model for the risk-adjusted dynamics of interest rates directly to the observed bond prices. However, as pointed out by various authors (e.g., [26, pp. 201–204]), once the risk-adjusted dynamics has been calibrated it is not obvious that such a model will tell us anything useful about the real-world dynamics of interest rates.

In principle, the risk neutral measure described by HJM models and the statistical measure describing the real dynamics of interest rates should be equivalent: a typical scenario simulated with a risk-neutral model could then be viewed as a possible, and hopefully typical, scenario for the future evolution of the yield curve. However, it turns to be difficult to account for a series of empirical facts observed in studies on the term structure of interest rates [5] in the framework of arbitrage-free term structure models. On one hand, it seems that the constraints implied by the absence of arbitrage in these models are so restrictive that the latter is obtained at the detriment of a correct representation of the dynamics of the yield curve [6]. For example, arbitrage-free interest rate models imply that long term rates never fall [12, 13] (i.e., increase almost-surely), a property which has no empirical counterpart i.e., fails to hold "almost-surely" under the real-world measure. On the other hand, some stylized empirical facts, such as the structure of principal components and the magnitudes of their variances, seem to have no theoretical counterpart in arbitrage-free (HJM) models: reproducing them requires introducing as many parameters as there are properties to be explained or postulating non-intuitive functional forms for the volatility of forward rates [21].

1.2. *Statistical properties of term structure movements*

As in any modeling approach, empirical observations should be the starting point in the construction of stochastic term structure models. We summarize below some important empirical facts about movements in interest rates (see [5, 6, 20, 27]):

1. Mean reversion: unlike stock prices, interest rates revert to a long term average. This behavior has resulted in models where interest rates are modeled as stationary processes [2, 31].
2. Smoothness in maturity: yield curves do not present highly irregular profiles with respect to maturity. Of course one could argue that with 50 or 60 data points it is difficult to assess the smoothness of a curve; this property should be viewed more as a requirement of market operators. A "jagged" yield curve would be considered as a peculiarity by any market operator. This is reflected in the practice of obtaining implied yield curves by smoothing data points using

splines. Note however that these interpolation/smoothing procedures are usually applied to the discount curve, i.e., to the function:

$$D(t,x) = \exp - \int_0^x r_t(u)\,du, \tag{1.1}$$

of which the forward rate curve is a logarithmic derivative, so the implied degree of smoothness for the forward curve itself is lower.

3. Irregularity in time: The time evolution of individual forward rates (with a fixed time to maturity) are very irregular. This should be contrasted with the regularity of forward rates with respect to time-to-maturity and reveals an *asymmetry* between the respective roles of the variables t and x.
4. Principal components: Principal component analysis of term structure deformations indicates that *at least* two factors of uncertainty are needed to model term structure deformations. In particular, forward rates of different maturities are imperfectly correlated. Empirical studies [5, 20] uncover the influence of a *level* factor which corresponds to parallel shifts of the yield curve, a *steepness* factor which corresponds to opposite changes in short and long term rates and a *curvature* factor which means that the curvature of the yield curve influences its evolution. More precisely, the third principal component, when projected on forward rates of different maturities, shows large coefficients maturities around one year and small coefficients on the two extremities of the yield curve [5]. Higher principal components show increasingly oscillating profiles in maturity and the variances associated to these principal components decay quickly [5, 20, 27]. The shapes of these principal components are stable across time periods and markets.
5. Humped term structure of volatility: Forward rates of different maturities are not equally variable. Their variability, as measured for example by the standard deviation of their daily variations, has a humped shape as a function of the maturity, with a maximum at $x \simeq 1$ year and decreases with maturity beyond one year [5]. This hump is always observed to be skewed towards smaller maturities. Moreover, though the observation of a single hump is quite common [21], multiple humps are never observed in the volatility term structure.

1.3. *Model dimensionality and its implications*

Another issue is the dimensionality of the model: the number of variables and the number of (independent) factors, which turns out to be more important for model properties than distributional assumption on interest rates [30]. Empirical term structure data consist of time series of interest rates of various maturities; for example, in the Eurodollar market interest rates of about 40 different maturities can be obtained on a daily basis. Instead of modeling the data as a time series of 40-component vectors, continuous-time interest rate models tend to represent the yield curve as a function of a *continuous* maturity variable T evolving with a continuous time parameter t. These models, all of which can be embedded in the HJM [4, 17] framework with the forward rate curve as the dynamical variable, take the initial term structure and the forward rate volatilities as inputs and model the evolution of forward rates $f(t,T)$ as an *infinite* family of scalar diffusions driven by a *finite* number of independent Wiener processes:

$$df(t,T) = \alpha(t,T)\,dt + \sum_{i=1}^{d} \sigma_i(t,T) \cdot dW_t^i. \tag{1.2}$$

This choice may appear surprising since it greatly reduces the number of degrees of freedom in the dynamics of the term structure. Given the disproportion between the number of independent noise sources and the number of variables, it also implies that if no constraint is imposed on the dynamics such a model will present obvious arbitrage opportunities: as shown by Heath, Jarrow & Morton [17], requiring the absence of arbitrage strategies involving bonds of all maturities imposes a strong restriction on the drift coefficient of the forward rates:

$$\alpha(t,T) = \sum_{i=1}^{d} \sigma(t,T) \int_t^T \sigma(t,u)\,du + \sum_{i=1}^{d} \sigma_i(t,T)\gamma_i(t), \tag{1.3}$$

where $\gamma_i(t)$ are predictable processes representing the difference between the risk-neutral and the actual dynamics. But it is not clear whether the constraint (1.3) has any econometric reality: when estimating models such as (1.2)–(1.3) empirically, standard practice is to add an independent "observational error" to each observed forward rate [23], these extra sources of randomness not being restriced by (1.3). The need for unconstrained error terms amounts to recognizing that (1.3) is not satisfied in the observations and that the effective number of sources of randomness is equal to the number of forward rates observed.

While this may be true, it is also true as noted above that the first three principal components of forward rate movements explain more than 95% of their variance, suggesting that a three factor model would be sufficient. Low dimensional factor models [11, 13] are usually justified by referring to such results. But this line of reasoning has a flaw: the covariance structure of the forward rates is not to be identified with the covariance structure of the driving factors. In fact, as we shall see in Sec. 3, an arbitrary number of independent factors can coexist with a small number of dominant principal components.

Also, while a low dimensional factor model might explain the variance of the forward rate itself, the same model may not be able to explain correctly the variance of the P&L of fixed income portfolios involving non-linear combinations of the same forward rates. In other words, a factor whose associated eigenvalue is small may have a non-negligible effect on the fluctuations of a fixed-income portfolio whose value may have a large sensitivity to this factor. This question is especially relevant when calculating quantiles and Value-at-Risk measures and pleads for including all factors present instead of retaining only the few dominant ones.

Finally, n-factor diffusion models also imply that a interest-rate contingent claim may be hedged with *any* set of $n+1$ (zero-coupon) bonds, the maturities of the hedging instruments not being linked to that of the position being hedged. By contrast, market practice is to hedge a interest-rate contingent payoff with bonds of the same maturity (unless, of course, liquidity considerations impose the trader to do otherwise). These practices reflect the existence of a risk specific to instruments of a given maturity; in economic terms this phenomenon is related to the relative segmentation in maturity of the fixed income market. This fact does not seem to have any theoretical counterpart in HJM models, where randomness in the movements of interest rates is not local in maturity but "distributed" over all maturities. The presence of a maturity-specific risk can be restored if, when taking the continuum limit, one also allows the number of sources of randomness to grow with the number of maturities, leading to a possibly infinite number of factors in the limit. This idea has a natural link to *random field* representations of term structure models [15, 19].

2. A Stochastic Evolution Equation for Term Structure Deformations

What are the lessons to be drawn from these empirical observations? We will first define some criteria which a model should try to respect in order to give a faithful statistical representation of interest rate fluctuations. Based on these considerations, we will then proceed to describe the deformations of the term structure by means of a *stochastic evolution equation*, translating each of the criteria outlined above into their mathematical equivalents.

2.1. *Parametrization of the forward rate curve*

We will parameterize the evolution of the term structure of interest rates by the instantaneous forward rate curve (FRC), denoted by $f_t(x)$ where the subscript t denotes time and $x \in [x_{\min}, x_{\max}]$:

$$f(t, T) = r_t(x), \quad x = T - t. \tag{2.1}$$

As remarked by Musiela [22], this parametrization has the advantage that the forward rate curve process r_t will belong to the same function space (a space of continuous curves defined on $[x_{\min}, x_{\max}]$) when t varies, which is not the case of the process $f(t, .)$ whose domain of definition $[t, t + x_{\max}]$ changes with time. This change of parametrization is therefore more than a mere convenience because it allows to choose a single state space for the forward curve process and represent the forward rate curve as a stochastic process in an appropriately chosen infinite-dimensional function space. Here $x_{\min}$ is the shortest maturity available on the market and $x_{\max}$ the longest. $r_t(x_{\min})$ will be called the *short rate*, $r_t(x_{\max})$ the long rate. The quantity $s(t) = r_t(x_{\max}) - r_t(x_{\min})$ is the *spread*.

In most interest rate models $x_{\min}$ is taken to be 0 and $x_{\max} = +\infty$ but this is not necessarily the best choice nor even realistic. First, it is obvious that in empirical applications maturities have a finite span and $x_{\max}$ will be typically 30 years or less depending on the applications considered. Second, the finiteness of $x_{\max}$ avoids some embarrassing mathematical problems related to the $x \to \infty$ limit [12, 13] which are not necessarily meaningful from an economic point of view. More importantly, we shall see that the $x_{\max} = +\infty$ limit is not "innocent": setting $x_{\max}$ to a large but finite value can be *qualitatively* different from taking it to be infinite.

2.2. *Decomposition of forward rate movements*

As mentioned above, given the exogenous, macroeconomic nature of the fluctuations in the short rate and the well known role of the short rate and the spread as two principal factors, we first proceed to "factor" them out of the model and parameterize the term structure as follows:

$$r_t(x) = r_t(x_{\min}) + s(t)[Y(x) + X_t(x)], \tag{2.2}$$

where Y is a deterministic *shape* function defining the average profile of the term structure and $X_t(x)$ a nonanticipative process describing the random deviations of the term structure from its long term average shape. With no loss of generality we require:

$$Y(x_{\min}) = 0, \quad Y(x_{\max}) = 1, \tag{2.3}$$

which results in

$$X_t(x_{\min}) = 0, \quad X_t(x_{\max}) = 0. \tag{2.4}$$

The process $X_t(x)$ then describes the fluctuations of a random curve with fixed endpoints. We will thus call X_t the deformation of the term structure at time t. Factoring out the fluctuations of the first two principal components then means modeling separately the process $(r_t(x_{\min}), s(t))$ and the deformation process $(X_t)_{t\geq 0}$.

Figure 1 shows the various configurations adopted by the deformation process $X_t(x)$ in the case of Eurodollar futures.

In a Gaussian framework, the uncorrelated nature of the principal component processes would entail their independence. In particular the first two principal components (which are roughly the spread and the short rate) would be independent from the deformation process X_t. However this assumption is not necessary in what follows and one can for example insert a state-dependent volatility or jump component in the short rate/spread process without altering the conclusions that follow.

2.3. *The short rate and the spread*

Among all interest rates, the short rate $r_t(x_{\min})$ is highly sensitive to the monetary policy of the central bank and cannot be considered as a market rate: its dynamics is exogenous to the fixed income market. One could therefore model its dynamics as an autonomous diffusion. However, as shown by Ait Sahalia [2], non-parametric tests tend to reject the diffusion hypothesis for $r_t(x_{\min})$ taken individually whereas the hypothesis of a bivariate diffusion for $(r_t(x_{\min}), s(t))$ is not rejected. Motivated by these remarks, we assume that the short rate and the long rate $(r_t(x_{\min}), r_t(x_{\max}))$ are described by a (bivariate) Markov process.

$$\begin{aligned} dr_t(x_{\min}) &= \mu_1(r_t(x_{\min}), s_t)\, dt + \sigma_{1,1}(r_t(x_{\min}), s_t)\, dZ_t^1 + \sigma_{1,2}(r_t(x_{\min}), s_t)\, dZ_t^2, \\ ds_t &= \mu_2(r_t(x_{\min}), s_t)\, dt + \sigma_{2,1}(r_t(x_{\min}), s_t)\, dZ_t^1 + \sigma_{2,2}(r_t(x_{\min}), s_t)\, dZ_t^2, \end{aligned} \tag{2.5}$$

where Z^1, Z^2 are two independent noise sources, which can be Wiener processes or more generally, may have discontinuous trajectories as observed in many empirical studies of short term interest rates. For example, the noise source in Eq. (2.5) could be replaced with a non-Gaussian Lévy process without modifying what follows. Examples of diffusion models of this type have been previously considered in the literature, see [7, 29].

2.4. *Term structure deformations as infinite-dimensional diffusions*

We are now left with the deformation process $(X_t)_{t\geq 0}$ to model. The first requirement we impose on X_t is its smoothness in maturity: at a given time t, X_t is a function defined on $[x_{\min}, x_{\max}]$ determined by the forward term structure which, as remarked above, is a "smooth" function of the time to maturity x. X_t should therefore belong to a suitable space H of smooth functions. In view of interpreting our results in terms of principal component analysis, we would like the state space H to have a Hilbert structure in order to define orthogonal projections of X_t onto a suitable basis of H.

The second requirement we impose is that X_t be a Markov process in H. This property, as remarked in [22], is already verified in the Heath-Jarrow-Morton [17] framework for the forward curve process f_t. Together with the hypothesis of continuity in time of the deformation process, this leads us to a diffusion model taking values in H. More precisely, stating that X_t is a H-valued diffusion process means that there exist a drift operator μ: $H \to H$ and a volatility operator σ: $H \to L_2(H)$, defined on H, such that the evolution of X_t is given by a stochastic differential equation in H:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, d\mathcal{B}_t, \tag{2.6}$$

where $\mathcal{B}_t$ is a Brownian motion taking values in H. Here μ and σ may be allowed to depend on the contemporaneous term structure — they can be a function of the whole curve $X_t(x), x \in [x_{\min}, x_{\max}]$.

Formally, Eq. (2.6) is a stochastic differential equation in an (infinite-dimensional) function space H. In order to give a proper meaning to Eq. (2.6), one should start by specifying the nature of the random noise source $\mathcal{B}_t$ such that the stochastic integral implicit in Eq. (2.6) can be properly defined. There are several ways to define a generalization of the Wiener process and a stochastic integral in an infinite dimensional space. The relation between these different constructions was clarified by [33] who showed that the natural setting for constructing infinite dimensional diffusions is a Hilbert space. Here we give a brief survey of the mathematical setting in order to make it accessible to readers familiar with stochastic differential equations in the finite dimensional setting. Given a (separable) Hilbert space H, such as $H = L^2([x_{\min}, x_{\max}], \nu)$ for some measure ν or an associated Sobolev space of smooth functions, and a complete orthonormal family $(e_n)_{n\geq 1}$ of H, the elements of H may be characterized by their decomposition on the basis $(e_n)_{n\geq 1}$:

$$\forall h \in H, \quad \exists!(h_n)_{n\geq 1}, \quad \sum_{n\geq 1} h_n^2 < +\infty, \quad \left| h - \sum_{n\geq 1}^{N} h_n e_n \right|_H \underset{N\to\infty}{\longrightarrow} 0. \tag{2.7}$$

One would then like to define a "standard" Brownian motion in H as being a superposition of independent scalar Wiener processes along each direction of the basis:

$$\mathcal{B}_t = \sum_{n\geq 1} W_n(t) e_n. \tag{2.8}$$

However, this sum does not converge in H since the coefficients in the expansion are not square summable: $\mathcal{B}_t$ cannot therefore be defined as a process taking values in H. However, for any unit vector $h \in H$ the "projection" of $\mathcal{B}_t$ on h defines a square integrable random variable:

$$\mathcal{B}_t(h) := \langle \mathcal{B}_t, h\rangle = \sum_{n\geq 1} W_n(t) h_n \in L^2(\Omega, H). \tag{2.9}$$

$\langle \mathcal{B}_t, h\rangle$ is in fact a scalar Brownian motion for each h and can be interpreted as the component of $\mathcal{B}_t$ along h. This property may be used to define $\mathcal{B}_t$ through its projections: one can define a *cylindrical Brownian motion* on H as a family $(\mathcal{B}_t)_{t\geq 0}$ of random linear functionals $\mathcal{B}_t$: $H \to R$ satisfying:

1. $\forall h \in H, \mathcal{B}_0(h) = 0$.
2. $\forall h \in H, \mathcal{B}_t(\phi)$ is an $\mathcal{F}_t$ — adapted scalar stochastic process.
3. $\forall h \in H - \{0\}, \frac{\mathcal{B}_t(\phi)}{|\phi|}$ is a one-dimensional Brownian motion.

In particular, if one takes any orthonormal basis (e_n) in H then its image $(W^n(t))_{t\geq 0} = (\mathcal{B}_t(e_n))_{t\geq 0}$ forms a sequence of independent standard Wiener processes in $\mathbf{R}$, which justifies the formal expansion (2.8). This property is independent of the choice of the orthonormal family (e_n). It is useful for building finite-dimensional approximations and is analogous to the construction of n-dimensional Brownian motion as a vector of n independent scalar Wiener processes.

Although the expansion (2.8) does not converge in H, it can be made to converge by weighting the terms with coefficients which decay with n:

$$W_t^Q = \sum_{n\geq 1} W_n(t) q_n e_n \in H, \quad q_n \geq 0 \ \sum_{n\geq 1} q_n^2 < \infty. \tag{2.10}$$

The projections of W^Q on the basis (e_n) still forms a sequence of independent Wiener processes, but the variances q_n^2 now decay with n. This construction is not invariant with respect to the choice of the basis $(e_n)_{n\geq 1}$: a "change of basis" will introduce correlations in the coordinates, with the covariances given by

$$\operatorname{cov}(\langle W_t^Q, h_1\rangle, \langle W_t^Q, h_2\rangle) = \langle Q \cdot h_1, h_2\rangle, \tag{2.11}$$

where Q is the symmetric positive operator defined by $Q \cdot e_n = q_n e_n$. W^Q is called a Q-Wiener process [9, 33]: it can be seen as a random field whose increments are uncorrelated in time but correlated in the x-direction.

The construction above can be carried out in any (separable) Hilbert space of functions: examples are weighted Sobolev spaces of real-valued functions on $[x_{\min}, x_{\max}]$, i.e., the space of functions $g \in L^2([x_{\min}, x_{\max}], \nu)$ such that the derivative $g^{(s)}$ of order s is also in $L^2([x_{\min}, x_{\max}], \nu)$. The Sobolev embedding theorem [1] then implies that if $s > k + 1/2$ then forward rate curves are k-times differentiable in the usual sense. Many recent papers have discussed other choices of state spaces for forward rate curves (see, e.g., [14]): a typical requirement in these discussions is that the forward rate is a continuous function of (time-to-)maturity so that point evaluation $x \mapsto r_t(x)$ is a continuous operation. Note however that the instantaneous forward rates $r_t(x)$ are *not* observable quantities: the only observable quantities are zero-coupon bond prices or LIBOR rates. These quantities are integral functionals of the forward rate curve, thus already continuous with respect to a weighted L^2 norm. In the following we shall take as our state space $H = L^2([x_{\min}, x_{\max}], \nu)$ and then give conditions under which the forward rate curve in our model actually belongs to Sobolev spaces $H^s([x_{\min}, x_{\max}], \nu)$ with a given smoothness $s \geq 1$. Due to the normalization used for the deformation process X_t, it actually verifies the boundary conditions (2.4) so we will see that $X_t \in H_0^s([x_{\min}, x_{\max}], \nu)$.

At this level of generality, not much can be said of the properties of the solutions of Eq. (2.6). In this section we will show how the description of term structure deformations through level, steepness and curvature of the yield curve reduces Eq. (2.6) to a stochastic partial differential equation, of which we study the simplest example.

2.5. *Market segmentation and local deformations*

As mentioned before, the maturity x is not simply a way to index different forward rates and instruments: the fact that fixed income instruments are ordered by maturity is important for market operators. For example, this is reflected in the hedging strategies of operators on the fixed income market: to hedge an interest rate risk of maturity $x = 8$ months, an operator will tend to use bonds (or other fixed-income instruments) of maturity close to 8 months: 6 months, 9 months. Although this strategy seems quite sensible, it does not correspond to the picture given by multifactor models: in a k-factor model, any $k + 1$ instruments can be used to hedge an interest rate contingent claim. For example in a two factor model one could use in principle a 30 year bond, a 10 year bond and a 6 year bond to hedge an instrument with maturity of two years! Needless to say, no sensible trader would follow such a strategy, which shows that in practice the factors which explain 95% of the variance are not enough to hedge 95% of the risk of an instrument with a non-linear payoff: this is precisely our principal motivations for introducing *maturity-specific* sources of randomness.

The existence of maturity-specific risk naturally leads to a market for such risk. Indeed, some macroeconomic theories of interest rates have considered the interest

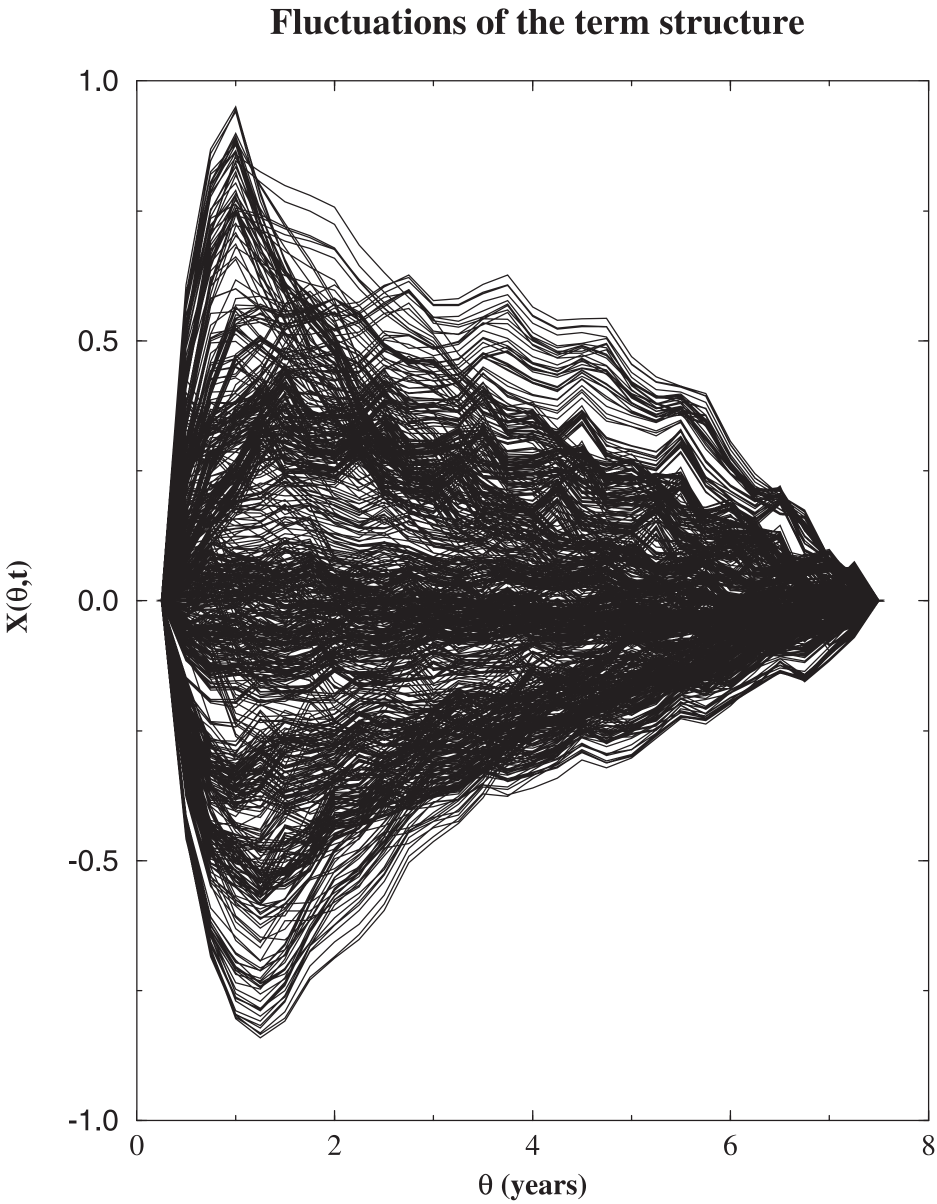

Fig. 1. Various configurations adopted by the function $X_t(x)$ defined in Eq. (2.2) as obtained from Eurodollar future contracts (1992–1997). This figure may be seen as a visualization of the stationary density of the process X_t.

rate market as being segmented: for example in a first approximation one can consider the market for US Treasury bills, Treasury notes and Treasury bonds as being 3 separate markets where prices are fixed independently. In a continuous-maturity model this would mean that the interest rate market is partitioned into independently evolving markets involving instruments with maturity between x and $x+dx$.

However this is not strictly true: as shown by principal component analysis, long-term rates *do* react to variations in the short rate in a way that is not explainable simply via parallel shifts and vertical dilations of the term structure. One way to conciliate the interdependence of rates of various maturities with the segmentation of markets across maturities is by considering deformations of the term structure that are *local* in maturity: a forward rate of maturity x is more sensitive to variations of rates with maturity close to x. We are not dealing here with a strict segmentation of the market into separately evolving markets but a "soft" segmentation which simply implies that the market for each maturity adjusts itself to the variation in rates of maturities immediately above and below it. This means for example that, among all rates of maturity $\geq$1 year, the 1 year rate will have a higher sensitivity and react more quickly to a variation in the short rate since it is closer in maturity.

2.6. *Level, steepness and curvature*

In mathematical terms, the local deformation hypothesis means that the variation of $X_t(x)$ will only depend on the behavior of $X_t(\cdot)$ around x. One can parameterize the shape of the term structure around a given maturity x by its first few derivatives: $X_t(x), \partial_x X_t, \partial_x^2 X_t, \ldots$ As noted before, empirical studies seem to identify the level of interest rates, the steepness (slope) of the term structure and its curvature as three significant parameters in the geometry of the yield curve [20]. In a market involving instruments of maturity between x and $x + dx$, these three features are described by the level of rates, and the first two derivatives with respect to x. Combining the local deformation hypothesis formulated in Sec. 5.1. with a local description of the term structure by level, steepness and curvature one obtains that the drift and volatility of $X_t(x)$ can only depend on $X_t(x)$, $\partial_x X_t$, $\partial_x^2 X_t$. Econometric evidence for the influence of the local slope and level on the movements of forward rates has been observed by [25] among others. Equation (2.6) then becomes a second order stochastic partial differential equation:

$$dX_t = \left[\frac{\partial X_t}{\partial x} + b\left(X_t(x), \frac{\partial X_t}{\partial x}, \frac{\partial^2 X_t}{\partial x^2}\right)\right] dt + \sigma\left(X_t(x), \frac{\partial X_t}{\partial x}, \frac{\partial^2 X_t}{\partial x^2}\right) dB_t(x),$$

$$\forall\, t \geq 0, \quad X_t(x_{\max}) = X_t(x_{\min}) = 0, \quad X_{t=0}(x) = X_0(x). \qquad (2.12)$$

This equation is the mathematical expression of the fact that deformations are local in maturity and that the deformation at maturity x depends on the level, steepness and curvature of the term structure around x.

As noted by [16, 22], HJM models also give rise to (first-order) stochastic PDEs for *risk-neutral* dynamics of the term structure. The new aspect in our approach is the inclusion of a second order term which captures the effect of curvature and changes the nature of the equation. In the general case where b and σ are nonlinear functions of their arguments, Equation (2.12) is not easy to study: indeed, it is not trivial to define properly what is meant by a solution of Eq. (2.12) and even less to study their regularity [9, 24]. In order to point out the differences with HJM-type models resulting from the local deformation hypothesis, we shall consider the case of a forward rate dynamics such as (1.2) which is perturbated by a term depending on the curvature:

$$dX_t = \left[\frac{\partial X}{\partial x} + b(t, x, X_t(x)) + \frac{\kappa}{2}\frac{\partial^2 X}{\partial x^2}\right] dt + \sigma(t, x, X_t(x))\, dB_t(x),$$
$$\forall\, t \geq 0,\;\; X_t(x_{\min}) = X_t(x_{\max}) = 0,\;\; X_{t=0}(x) = X_0(x). \quad (2.13)$$

Mathematical properties of stochastic PDEs such as Eq. (2.13) have been widely studied in the mathematical literature: see [9, 18, 24, 32]. The approach adopted here is that of [9]. In the following section we study the simplest case where volatility is constant; surprisingly, we will show that this simple case already presents many of the desirable features enumerated in Sec. 1.

3. The Linear Parabolic Case

In order to illustrate what are the type of dynamics implied by Eq. (2.13) for term structure deformations, we will now study the simplest example of the above equations which incorporates the influence of local steepness and curvature, namely the case where σ is independent of X_t. For the sake of simplicity we will deal here with the constant volatility case but all the results below remain valid in the case of an arbitrary deterministic function of time t. The case of constant volatility leads us to the following stochastic partial differential equation:

$$\frac{\partial X}{\partial t} = \left[\frac{\partial X}{\partial x} + \frac{\kappa}{2}\frac{\partial^2 X}{\partial x^2}\right] dt + \sigma_0\, dB_t(x), \quad (3.1)$$

$$\forall\, t \geq 0, \quad X_t(x_{\min}) = X_t(x_{\max}) = 0, \quad (3.2)$$

$$\forall\, x \in [x_{\min}, x_{\max}], \quad X_{t=0}(x) = X_0(x). \quad (3.3)$$

3.1. *Eigenmodes and principal components*

Let $x^* = x_{\max} - x_{\min}$ be the maturity span of the observed forward rate curve. By translating the maturity variable one can assume $x_{\min} = 0$ without loss of generality in what follows. We consider as state space for our solutions the Hilbert space H of real-valued functions defined on $[0, x^*]$ with the scalar product:

$$\langle f, g\rangle_H = \int_0^{x^*} dx \exp\left(\frac{2x}{\kappa}\right) f(x)\, g(x). \quad (3.4)$$

The subscript H in $\langle ., .\rangle_H$ will be omitted in most of this section. Let A be the operator in H defined by:

$$A \cdot u = \frac{\partial u}{\partial x} + \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2}. \quad (3.5)$$

It is not difficult to show that A has a discrete spectrum, with eigenvalues and eigenfunctions given by:

$$A \cdot e_n = -\lambda_n e_n, \quad (3.6)$$

$$\lambda_n = \frac{1}{2\kappa}\left(1 + \frac{n^2\pi^2\kappa^2}{x^{*2}}\right), \tag{3.7}$$

$$e_n(x) = \sqrt{\frac{2}{x^*}}\sin\left(\frac{nx\pi}{x^*}\right)\exp\left(-\frac{x}{\kappa}\right), \tag{3.8}$$

where n takes all integer values ≥ 1. Here the eigenfunctions e_n have been normalized such that $(e_n)_{n\geq 1}$ is an orthonormal basis of H.

$$\langle e_n, e_m \rangle = \delta_{nm}. \tag{3.9}$$

The functions (eigenmodes) $e_n(x)$ play the role of the principal components for the deformation process X_t. That is, if we perform a principal component analysis on a realization of the process X_t, for a large enough sample the empirical principal components would reproduce the eigenmodes $e_n(x)$. The first two of these eigenmodes are shown in Fig. 2. The role of the exponential term in Eq. (3.8) is clearly visible: the eigenfunctions become "skewed" towards shorter maturities and only a single hump, whose position is determined by the value of κ, is visible. Recall that this exponential term stems simply from the fact that we are parameterizing the forward rate process by time to maturity x instead of maturity date T [22]. In particular, in contrast with multifactor models [21], there is no need to use a complicated volatility structure $\sigma(t, x)$ to obtain a volatility hump. The position of the hump gives a (first) simple method for estimating the value of κ from empirical observations. Remark also that the exponential factor prevents more than one hump from appearing in the eigenfunctions: as a result, multi-humped deformations are not observed, in concordance with the empirical observations invoked in Sec. 2.

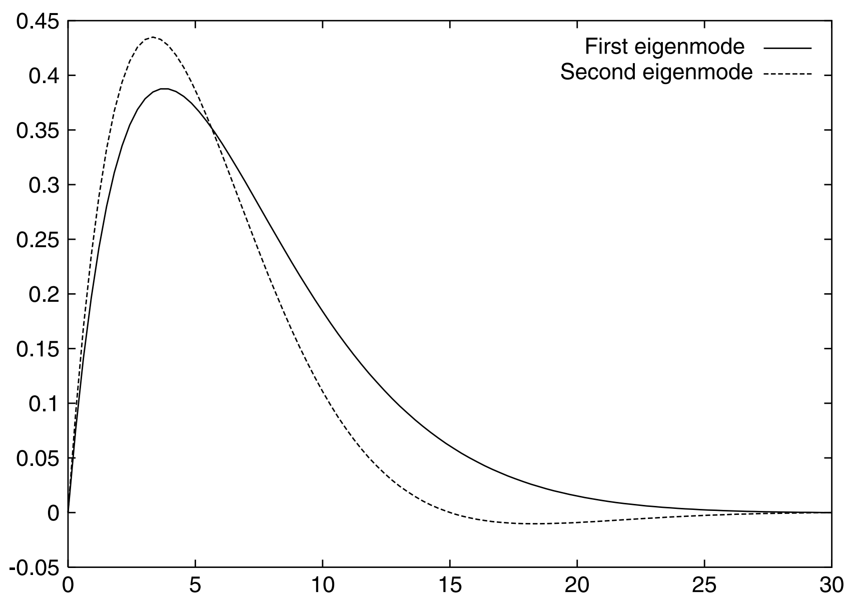

Fig. 2. First two eigenmodes of the operator A, with $\kappa = 4$ and $x^* = 30$ years. Note the maxima situated at short maturities.

The Green function (propagator) associated to the operator A may then be expressed in terms of an eigenmode expansion:

$$G(t,x,y) = \sum_{n\geq 1} \exp(-\lambda_n t) e_n(x) e_n(y). \tag{3.10}$$

The random field $X_t(x)$ is then said to be a *mild solution* [9] of Eq. (3.1) if it verifies the equation in the following integral form:

$$X_t(x) = \int_0^{x^*} G(t,x,y) X_0(y)\, dy + \int_0^t ds \int_0^{x^*} G(t-s,x,y)\sigma_0 d\mathcal{B}_s(y). \tag{3.11}$$

Let $X_t(x)$ be the solution of Eq. (3.11). Define the coordinates of the solution in the eigenvector basis as:

$$x_n(t) = \langle X_t, e_n\rangle = \int_0^{x^*} dx\, X_t(x) e_n(x) \exp\left(\frac{2x}{\kappa}\right). \tag{3.12}$$

The coefficients $x_n(t)$ therefore represents the projection of the deformation process X_t on the nth principal component (eigenmode). For each n, $x_n(t)$ is then a solution of a linear stochastic differential equation:

$$dx_n(t) = -\lambda_n x_n(t)\, dt + \sigma_0\, dW_t^n, \tag{3.13}$$

where $W_t^n = \mathcal{B}_t(e_n)$ are independent standard Wiener processes. $(x_n(t))_{n\geq 1}$ therefore constitute a sequence of independent Ornstein–Uhlenbeck processes. The Ornstein–Uhlenbeck process is the process used to represent the short rate in the Vasicek model [31]: it possesses the fundamental property of *mean reversion* which has made it a popular model in interest-rate modeling. In our case, this mean reversion is observed in the principal components of the yield curve: while individual forward rates may have a non-stationary and irregular behavior the yield curve as a whole will converge to a stationary state with a mean-reverting behavior. We can thus represent the random field $X_t(x)$ as an infinite factor model where each factor follows an Ornstein–Uhlenbeck process:

$$X_t(x) = \sum_{n\geq 1} x_n(t) e_n(x), \tag{3.14}$$

$$x_n(t) = e^{-\lambda_n t}\langle e_n, X_0\rangle + \int_0^t ds\, e^{-\lambda_n (t-s)} \sigma_0 dW_s^n. \tag{3.15}$$

Equation (3.17) has an interesting interpretation. Remember that $x_n(t)$ the projection of the deformation process X_t on the nth principal component. Equation (3.15) expresses $x_n(t)$ as the sum of two components, the first one being the contribution of the initial term structure to x_n and the second one its stationary value. Equation (3.15) may then be interpreted by stating that the forward rate curve “forgets” the contribution of the nth principal component to the initial term structure at an exponential rate with characteristic time

$$\tau_n = \frac{1}{\lambda_n} = \frac{2\kappa}{1 + \frac{n^2\pi^2\kappa^2}{x^{*2}}}. \tag{3.16}$$

Therefore, a perturbation of the initial term structure due to the nth principal component will disappear or be smoothed out after a typical time τ_n which decreases with n: perturbations which have an irregular profile in maturity die out more quickly than smoother ones. This property guarantees that in the long term only smooth deformations of the term structure remain visible, which guarantees the smoothness in maturity and shows the important relation between the smoothing property of the operator A and the decay of its eigenvalues: an operator with a quickly decaying spectrum will guarantee a fast decay (in time) of the singularities appearing in the term structure and restore smoothness in maturity. This gives a second interpretation of the parameter κ: in addition to determining the position of the volatility hump, it also determines the decay rate of perturbations of the term structure. This interpretation gives a second, independent method for estimating the model parameters from empirical data. One can also construct more general models where these two roles of κ can be attributed to two separate parameters by introducing a term structure $\kappa(x)$ [8].

The factor representation (3.14) can be easily generalized to the case where factors have unequal volatilities (σ_n, $n \geq 1$):

$$x_n(t) = e^{-\lambda_n t}\langle e_n, X_0\rangle + \int_0^t ds\, e^{-\lambda_n(t-s)}\sigma_n\, dW_s^n. \tag{3.17}$$

X_t is then the (mild) solution of

$$dX_t = A \cdot X_t\, dt + \sigma \cdot\, d\mathcal{B}_t, \tag{3.18}$$

where σ is now an operator verifying

$$\sigma \cdot e_n = \sigma_n^2 e_n, \quad \sigma \cdot A = A \cdot \sigma. \tag{3.19}$$

Given the explicit form of λ_n, it is easily verified that the series

$$E[X_t(x)] = \sum_{n\geq 1} e^{-\lambda_n t}\langle X_0, e_n\rangle e_n(x), \tag{3.20}$$

$$\mathrm{Var}[X_t(x)] = \sum_{n\geq 1} \sigma_n^2 \frac{1-\exp(-2\lambda_n t)}{2\lambda_n} e_n(x)^2, \tag{3.21}$$

are absolutely convergent for all (t, x) and (3.14) defines a Gaussian random field with mean and variance given by (3.20)–(3.21).

3.2. *Average term structure and mean reversion*

Under the above assumptions, one can calculate the average shape of the term structure of forward rates from Eq. (2.2).

$$E[r_t(x)] = E[r_t(x_{\min})] + E[s(t)]Y(x). \tag{3.22}$$

The shape function $Y(x)$ can therefore be chosen in order to reproduce the average term structure. Bouchaud *et al.* [5] propose the following shape function:

$$Y(x) = \sqrt{\frac{x}{x*}}, \tag{3.23}$$

which appears to give a good fit of the average shape of the Eurodollar term structure for maturities ranging from 3 months to 10 years [5]. However, the precise analytic form of the shape function Y does not affect the results above.

How does the yield curve fluctuate around its average shape? It is easily seen from Eq. (3.21) that the process X_t converges for $t \to \infty$ to a Gaussian random field X_∞ with mean zero and covariance:

$$\mathrm{Cov}(X_t(x), X_{t'}(x')) = \sum_{n\geq 1} \sigma_n^2 \frac{e_n(x)e_n(x')e^{-\lambda_n(t-t')}}{2\lambda_n}. \qquad (3.24)$$

Viewed as a process in H, (X_t) is an Ornstein Uhlenbeck process whose stationary distribution is a Gaussian distribution with covariance

$$C = \frac{1}{2}\sigma \cdot \sigma^* A^{-1}, \quad \text{with spectral decomposition } C \cdot e_n = \frac{\sigma_n^2}{2\lambda_n} e_n, \quad n \geq 1. \qquad (3.25)$$

An important consequence of Eq. (3.25) is that the variance of the principal components can be small without the factor volatilities being small: a fast increase of λ_n has the effect of damping out the variance of the higher principal components. In terms of term structure movements, stationarity of term structure deformations implies a mean-reverting behavior of forward rates. The analytic form of the stationary measure further allows to compute the probability of a given yield curve deformation by decomposing it on the principal component basis.

3.3. *Smoothness in maturity*

As mentioned above, an important property of forward curves is their smoothness in maturity x, as opposed to their irregular behavior in t. In the parabolic SPDE model, this property is guaranteed by the smoothing effect of the second order derivative in x. Moreover, one can precisely relate the rate of decay of volatilities σ_n of the factors to the smoothness of the forward rate curve:

Proposition 3.1. *If the volatilities σ_n decay faster than $n^{-\beta}$ i.e., $(n^\beta \sigma_n)_{n\geq 1}$ is bounded then, with probability* 1, *the stationary solution of* (3.18)–(3.19) *is k times differentiable with respect to the maturity variable x for every $k < \beta$.*

Proof. Let P_∞ be the stationary distribution of X_t on $L^2([x_{\min}, x_{\max}], \nu)$. P_∞ is a Gaussian measure with covariance C where $C = A^{-1} \cdot \sigma^* \sigma/2$. So $C \cdot e_n = \sigma_n^2/2\lambda_n$. Now assume $\sigma_n = a_n n^{-\beta}$ with (a_n) bounded and take $k < \beta$. Then, $E^{P_\infty}|\langle X_t, e_n\rangle|^2 = \sigma_n^2/2\lambda_n$. Now take any real number $s > k + 1/2$. The Sobolev space $H^s([x_{\min}, x_{\max}])$ is defined by

$$h \in H^s \Leftrightarrow h \in L^2([x_{\min}, x_{\max}], \nu), \quad |h|_{H^s}^2 = \sum_{n\geq 1} n^{2s} |\langle h, e_n\rangle|^2 < +\infty.$$

Substituting $h = X_t$ and taking expectations one obtains

$$E^{P_\infty}\|X_t\|_{H^s}^2 = \sum_{n\geq 1} \frac{n^{2s}\sigma^2}{2\lambda_n} = \sum_{n\geq 1} \frac{a_n n^{2s-2\beta}}{2\lambda_n}.$$

Since λ_n increases as n^2, the summands are bounded by a sequence decaying as $n^{2s-2-2\beta}$. So $2s-2-2\beta<-1$ and the series is absolutely convergent:

$$E^{P_\infty}\|X_t\|^2_{H^s}<+\infty.$$

Therefore $P_\infty(\|X_t\|^2_{H^s}<+\infty)=1$ so $\forall\, s>k+\frac{1}{2}$, $P_\infty(X_t\in H^s)=1$. By the Sobolev embedding theorem [1], every element of $H^s, s>k+1/2$ is k times differentiable with respect to x hence the result. □

This result means that one can then read off the degree of smoothness of the forward rate curve from the decay rate of factor volatilities, in a very simple way. In particular, if the number of driving factors is finite, the forward rate curve is infinitely differentiable in x.

It also implies that we can read off the degree of differentiability with respect to maturity from the rate of decay of variances in the principal component analysis: if the variances decay as $n^{-\beta}$, then the maximal degree of differentiability of forward rate curves is less than β. Empirical studies [5] indicate a rate of decay between $\beta=2$ and $\beta=3$. However β is the asymptotic rate of logarithmic decay and as such may be difficult to determine precisely from empirical data since by definition one has observations on a discrete tenor of maturities. This point is also discussed by [10].

Note finally that, even in the case of flat term structure where $\sigma=\sigma_0 I$, in which case the noise term is space-time white noise, the dependence in maturity x is still twice as smooth as in t: the asymmetry between t and x dependence is not due to an ad-hoc choice of volatility structure [21] or of the noise source (as [28], see below) but is a generic property due to the presence in (3.1) of the second derivative with respect to maturity.

3.4. *Random strings: parabolic vs. hyperbolic formulation*

In a recent work [28], it has been proposed to consider stochastic partial differential equations of *hyperbolic* type to describe the evolution of the forward rate curve. Such an equation differs from the above one through the presence of a second-order time derivative which dominates the dynamics. The hyperbolic model proposed by [28] is formulated in terms of the forward rate process itself (in the spirit of the HJM model [17]) and not in terms of the deformation process X_t, by examining the effect of a second-order time derivative in the equations of Sec. 5. Let us therefore consider the general case where the evolution equation contains both a propagation term and a diffusion term:

$$f\frac{\partial^2 X}{\partial t^2}+\frac{\partial X}{\partial t}=\frac{\partial X}{\partial x}+\frac{\kappa}{2}\frac{\partial^2 X}{\partial x^2}+\sigma\cdot dB_t(x), \tag{3.26}$$

$$\forall\, t\geq 0,\quad X_t(0)=X_t(x^*)=0, \tag{3.27}$$

$$X_{t=0}(x)=X_0(x),\quad \frac{\partial X_{t=0}}{\partial t}=Y_0. \tag{3.28}$$

The above equation is analogous to that of a vibrating elastic string, hence the name of "string models" given to such descriptions of term structure movements. The case $f = 0$ is the one studied in Sec. 5; the case $f \to \infty$ (the other parameters being appropriately rescaled) is the stochastic wave equation [9], the "space" variable being $x + t$. Note that the stochastic PDE proposed by [28] is formulated as a PDE perturbated by a two-parameter ("space-time") noise (called a "stochastic string shock") while our Eq. (3.1) or the general case Eq. (2.12) was presented above as an evolution equation for a curve in some function space. In the case of a parabolic SPDE, where only the first derivative with respect to time is involved, the two approaches are equivalent for a two-parameter process. Choosing one approach or the other then amounts to viewing the solution of a stochastic PDE either as a random field or as a stochastic process in a function space. Implicit in this choice is whether the object of interest for modeling purposes is an individual interest rate or the deformation of a multivariate object, namely the term structure. The parabolic equation in Sec. 3 has the merit of emphasizing the asymmetric roles of the variables x and t, an empirically desirable feature which is not present, as we shall see below, in the hyperbolic case.

Manifestly, the operator on the right hand side of Eq. (3.26) is the same as in Eq. (3.1). This means that the deformation eigenmodes (the eigenfunctions of A) will remain the same as in Eq. (3.1) studied above but the projection of the process X_t on each of them will be different. For example, a stationary solution of Eq. (3.26) will not give the same weight to the eigenmodes as in the parabolic case studied in Sec. 5 and therefore the results of a Principal Component Analysis of Eq. (3.26) will differ from that of Eq. (3.1) in terms of the eigenvalues.

The representation of the equation in terms of its projections on the eigenmode basis e_n gives as above a stochastic equation for the scalar process $y_n(t) = \langle X_t, e_n \rangle$:

$$f \frac{d^2 y_n}{dt^2} + \frac{dy_n}{dt} = -\lambda_n y_n(t) + \sigma_0 \dot{W}_t^n. \tag{3.29}$$

This formal second-order stochastic differential equation is interpreted in the usual way, as follows. Consider the Green function $U_n(t)$ of the operator $f \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \lambda_n u$, given by:

$$U_n(t) = \frac{1}{f\sqrt{1 - 4\lambda_n f}} [e^{r_{1,n}t} - e^{r_{2,n}t}] 1_{t>0}, \tag{3.30}$$

where $r_{1,n}$ and $r_{2,n}$ being the roots of the associated characteristic equation:

$$fr^2 + r + \lambda_n = 0. \tag{3.31}$$

The process $x_n(t)$ is then given by the stochastic convolution integral:

$$y_n(t) = a_n e^{-r_{1,n}t} + b_n e^{-r_{2,n}t} + \int U_n(t-s)\, dW_s^n, \tag{3.32}$$

which is well-defined since the integrator is square-integrable in s. Here a_n and b_n are defined by the initial conditions. Depending on the values of f and κ, two scenarios are possible:

1. Oscillatory behavior: if

$$f > \frac{\kappa}{2\left(1 + \frac{\pi^2\kappa^2}{x^{*2}}\right)}, \tag{3.33}$$

then for all $n \geq 1$, Eq. (3.31) has two complex conjugate roots given by:

$$r_{1,n} = \frac{i\sqrt{4\lambda_n f - 1} - 1}{2f} = -\frac{1}{2f} + i\omega_n, \tag{3.34}$$

$$r_{2,n} = \frac{-i\sqrt{4\lambda_n f - 1} - 1}{2f} = -\frac{1}{2f} - i\omega_n. \tag{3.35}$$

The real part gives an exponential damping of the initial conditions which characterizes the mean reverting behavior of X_t as in the parabolic case. First remark that, unlike the parabolic case where "bumpy" principal components which contribute the most to non-smoothness in maturity decay more quickly, here all principal components decay with the same speed, i.e., a mean reversion time of $2f$. Recall that κ still determines the position of the volatility hump so $\kappa \simeq 1$ year. So (3.33) implies that $f > 6$ months. The mean reversion time of the whole curve is thus around a year. However, a new phenomenon appears: the principal components do not simply revert to their mean but *oscillate* around their mean with a frequency $\omega_n/2\pi$ which increases with n:

$$x_n(t) = A_n e^{-t/2f}\cos(\omega_n t + \phi_n). \tag{3.36}$$

The phase ϕ_n and amplitude factor A_n are determined by (two) initial conditions (see below). The oscillation of the term structure around its mean is not necessarily an undesirable feature of this model and indeed can be justified on economic grounds [8]. But the slow mean reversion combined with increasingly faster oscillations of the higher order principal components leads to non-smoothness in maturity of the solutions of Eq. (3.26) [9]: in fact one should expect the cross-sections in time or maturity to have the same irregularity which, as pointed out in Sec. 1, is not a desirable feature for a term structure model.

2. Selective damping of principal components: if

$$f < \frac{\kappa}{2\left(1 + \frac{\pi^2\kappa^2}{x^{*2}}\right)}, \tag{3.37}$$

then $\exists N > 1$ such that for $n \leq N$, Eq. (3.31) has two real, negative roots whereas for $n > N$ the roots are complex conjugates with negative real parts. The projections of the deformations process X_t on the first N eigenmodes will have a mean reverting behavior as in the parabolic case,[1] with a mean reversion time increasing with n. For $n > N$, $x_n(t)$ will have a damped oscillatory behavior,

[1]In fact the decay of the initial condition is described in this case by the superposition of two decreasing exponentials with time constants given by r_{1n}^{-1} and r_{2n}^{-1}.

with a damping time $\tau = 2f$ independent of n and an oscillation frequency increasing with n as above.

Another crucial difference between Eq. (3.26) and Eq. (3.1) is the nature of the initial conditions. In the case of the parabolic equation (3.1) the problem has a well defined solution once the initial term structure is specified through X_0. This is not sufficient in the case of Eq. (3.26): one must also specify the derivative with respect to time at $t = 0$. In the case of a vibrating string, this means specifying the initial position and the initial velocity of each point of the string. For a model of the forward rate curve, this can be inconvenient: while the initial term structure is the natural input for the initial condition of a dynamic model, the time derivative of the forward rates is not easily evaluated, especially given the irregularity in time of forward rate trajectories which prevents such a model from being calibrated in a numerically stable manner.

We therefore conclude that the question of including a second-order time derivative in Eq. (3.26) is not simply a matter of taste: its radically changes both the dynamic properties of the equation and the nature of the initial conditions needed to calibrate the model, in an empirically undesirable fashion. Our analysis thus pleads for a description of yield curve deformations through a parabolic rather than hyperbolic SPDE.

4. Conclusion and Perspectives

We have presented a parsimonious stochastic model for describing the fluctuations of the term structure of forward rates: the forward rate curve is described as a random curve oscillating around its long term average or, alternatively, as random field solution of a stochastic PDE. The model studied in Sec. 3 should be viewed as the simplest example of model incorporating the effect of slope and curvature in forward rate dynamics. However this simple example has the benefit of emphasizing the role of the second derivative with respect to maturity in the evolution of the term structure: indeed, as we have seen above, it is this second derivative which tames the maturity-specific randomness and maintains a regularity in x while allowing for independent shocks along maturities. It also gives the correct form for the principal components as well as a qualitatively correct estimate for their associated eigenvalues. These results show the importance of the concept of local deformation explained in Sec. 2.5, of which our equation is the simplest example. The model in Sec. 3 can be generalized to the case of state and time-dependent coefficients [8] but the point is that a linear model with constant coefficients and flat term structure of volatility is already sufficient for reproducing empirical facts which, in standard interest rate models, need fine tuning of many parameters. Let us recall our main results:

1. The linear parabolic SPDE (3.1) reproduces the qualitative form of the principal components, the presence of a humped volatility term structure and the mean

reverting properties of forward rates with *two* parameters and without inserting ad-hoc structures in the noise terms.
2. The term structure deformation solution of (3.1) may be represented as a Gaussian random field $X_t(x)$, as an infinite-factor model (3.14) with Ornstein-Uhlenbeck dynamics for the factors or as a diffusion X_t in a space of curves.
3. A large (or infinite) number of factors is compatible with the fact that a few principal components capture most of the variance. The decay in the variance of principal components results from the smoothing effect of the evolution operator and is due to the impact of the *curvature* (convexity) on forward rate dynamics.
4. We do not require a humped term structure for the diffusion coefficient σ in order to obtain a humped term structure for the volatilities of the forward rates: in fact a *flat* term structure for σ already generates this hump in standard deviations of forward rate movements.
5. The model provides a link between the *geometry* of the forward rate curve — its shape and smoothness, measures by the principal components and their variances — and the *dynamics*, i.e., the time series properties of forward rates, as measures by their mean reversion time and autocorrelation properties. Both sets of properties are governed by the same parameters.

As mentioned in the introduction, our objective has been to obtain a faithful continuous-time representation of the statistical properties of the forward rate curve. Obviously such a model can be useful for simulating realistic scenarios for yield curve movements. It remains to establish the link with the arbitrage pricing approach and examine the consequences of our model for hedging interest rate options. As shown by [4], such an analysis requires a careful specification of the class of strategies one is willing to consider; also, as noted by [5], taking into account transaction costs eliminates theoretical “arbitrage opportunities” present in a model such as (3.1). These issues remain to be explored in more detail.

A remaining question is to explain the economic origin for the second-order term in (3.1). We discuss this issue in a forthcoming work [8].

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