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OPTION PRICING MODELS WITH JUMPS: INTEGRO-DIFFERENTIAL EQUATIONS AND INVERSE PROBLEMS.

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Abstract. *Observation of sudden, large movements in the prices of financial assets has led to the use of stochastic processes with discontinuous trajectories – jump processes – as models for financial assets. Exponential Lévy models provide an analytically tractable subclass of models with jumps and the flexibility in choice of the Lévy process allows to calibrate the model to market prices of options and reproduce a wide variety of implied volatility skews/smiles.*

We discuss the characterization of prices of European and barrier options in exponential Lévy models in terms of solutions of partial integro-differential equations (PIDEs). These equations involve, in addition to a second-order differential operator, a non-local integral term which requires specific treatment both at the theoretical and numerical level. The study of regularity of option prices in such models shows that, unlike the diffusion case, option price can exhibit lack of smoothness. The proper relation between option prices and PIDEs is then expressed using the notion of viscosity solution. Numerical solution of the PIDE allows efficient computation of option prices.

The identification of exponential Lévy models from option prices leads to an inverse problem for such PIDEs. We describe a regularization method based on relative entropy and its numerical implementation. This inversion algorithm, which allows to extract an implied Lévy measure from a set of option prices, is illustrated by numerical examples.

1 INTRODUCTION

The shortcomings of diffusion models in representing the risk related to large market movements have led to the development of various option pricing models with jumps, where large returns are represented as discontinuities in prices as a function of time. Models with jumps allow for a more realistic representation of price dynamics and a greater flexibility in modelling and have been the focus of much recent work [11].

Exponential Lévy models, where the market price of an asset is represented as the exponential $S_t = \exp(rt + X_t)$ of a Lévy process X_t, offer analytically tractable examples of positive jump processes which are simple enough to allow a detailed study both in terms of statistical properties and as models for risk-neutral dynamics i.e. option pricing models. Option pricing with exponential Lévy models is discussed in [11, 17, 25, 36]. The flexibility of choice of the Lévy process X allows to calibrate the model to market prices of options and reproduce a wide variety of implied volatility skews/smiles [12]. The Markov property of the price allows to express prices of European and barrier options in terms of solutions of partial integro-differential equations (PIDEs) with a non-local integral term which requires specific treatment both at the theoretical and numerical level [15]. Deterministic computational methods can then be used to compute option prices [14, 27].

We present here a summary of our recent work on partial integro-differential equations (PIDEs) for option pricing in exponential Lévy models and related inverse problems arising in model calibration. Exponential Lévy models are presented in section 2. Under some regularity conditions, the value of European and barrier options can be represented in terms of solutions to an integro-differential equations (section 3). However, in section 4 we show that such regularity conditions fail to hold in many models used in finance. This prompts us to take a closer look, in section 5, at the relation between option prices using the notion of viscosity solution. The calibration of such models to market prices requires extracting the model parameters – the Lévy measure and the diffusion coefficient – from a set of observed option prices. We discuss this ill-posed inverse problem in section 6 and propose a regularization scheme based on penalization by relative entropy, which enables to construct an arbitrage–free risk–neutral exponential Lévy models compatible with a given set of option prices. Section 7 outlines some directions for future research.

2 LEVY PROCESSES AND EXPONENTIAL LEVY MODELS

We consider here a class of discontinuous stochastic models in which the risk neutral dynamics of the underlying asset is given by $S_t = \exp(rt+X_t)$ where X_t is a Lévy process: a process with independent, stationary increments with discontinuous trajectories.

2.1 Lévy processes: definitions

A Lévy process is a stochastic process X_t with stationary independent increments which is continuous in probability (but may have discontinuous trajectories). Without loss of generality we assume $X_0 = 0$. The characteristic function of X_t has the following

Lévy-Khinchin representation [34]:

$$E[e^{izX_t}] = \exp t\phi(z) \qquad \phi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|\leq 1})\nu(dx),$$

where $\sigma \geq 0$ and γ are real constants and ν is a positive Radon measure on $\mathbb{R} - \{0\}$ verifying

$$\int_{-1}^{+1} x^2 \nu(dx) < \infty, \qquad \int_{|x|>1} \nu(dx) < \infty.$$

The random process X can be interpreted as the independent superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes with various jump sizes x, $\nu(dx)$ being the intensity of jumps of size x.

In general ν is not a finite measure: $\int \nu(dx)$ need not be finite. In the case where $\lambda = \int \nu(dx) < +\infty$, the measure ν can be normalized to define a *probability measure* μ which can now be interpreted as the distribution of jump sizes:

$$\mu(dx) \quad = \quad \frac{\nu(dx)}{\lambda}.$$

The jumps of X are then described by a *compound Poisson* process with λ as jump intensity (average number of jumps per unit time) and jump size distribution $\mu(.)$. More generally, if $\int |x|\nu(dx) < \infty$, the (possibly infinite) sum of jumps is absolutely convergent with probability 1 and X_t can be represented as a pathwise sum of a Brownian motion plus jumps:

$$X_t \quad = \quad \sigma W_t + \gamma_0 t + \sum_{0<s\leq t} \Delta X_t \tag{1}$$

where $\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx)$. In this case the compensation of small jumps is not needed and the Lévy-Khinchin representation reduces to:

$$\phi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma_0 z + \int_{-\infty}^{\infty} (e^{izx} - 1)\nu(dx).$$

In the case where $\int |x|\nu(dx) = \infty$ the jumps have infinite variation and small jumps need to be compensated.

A Lévy process is a (strong) Markov process; the associated semigroup is a convolution semigroup and its infinitesimal generator $L^X : f \rightarrow L^X f$ is an integro-differential operator given by:

$$\begin{aligned} L^X f(x) \;&=\; \lim_{t\rightarrow 0} \frac{E[f(x+X_t)] - f(x)}{t} \;= \\ &=\; \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \gamma\frac{\partial f}{\partial x} + \int \nu(dy)[f(x+y) - f(x) - y1_{\{|y|\leq 1\}}\frac{\partial f}{\partial x}(x)] \end{aligned} \tag{2}$$

which is well defined for $f \in C^2(\mathbb{R})$ with compact support.

2.2 Exponential Lévy models

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modelled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. $\mathcal{F}_t$ is usually taken to be the price history up to t. Under the hypothesis of absence of arbitrage there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted prices of all financial assets are $\mathbb{Q}$- martingales; in particular the discounted underlying $(e^{-rt}S_t)$ is a martingale under $\mathbb{Q}$.

In *exponential Lévy* models, the (risk-neutral) dynamics of S_t under $\mathbb{Q}$ is represented as the exponential of a Lévy process:

$$S_t = S_0 e^{rt+X_t}.$$

Here X_t is a Lévy process (under $\mathbb{Q}$) with characteristic triplet (σ,γ,ν), and the interest rate r is included for ease of notation. The absence of arbitrage then imposes that $\hat{S}_t = S_t e^{-rt} = \exp X_t$ is a martingale, which is equivalent to the following conditions on the triplet (σ,γ,ν):

$$\int_{|y|>1} \nu(dy)e^y < \infty \quad , \quad \gamma = \gamma(\sigma,\nu) = -\frac{\sigma^2}{2} - \int (e^y - 1 - y 1_{|y|\leq 1})\nu(dy). \tag{4}$$

We will assume this relation in the sequel. The infinitesimal generator L^X then becomes:

$$L^X f(x) \;=\; \frac{\sigma^2}{2}\left[\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right] + \int_{-\infty}^{\infty} \nu(dy)\;\; \left[f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}(x)\right].$$

We will also use the notation $Y_t = rt + X_t$. Y_t is then a strong Markov process with infinitesimal generator

$$Lf \;=\; L^X f + r\frac{\partial f}{\partial x}. \tag{5}$$

While in principle one can have both a non-zero diffusion component $\sigma \neq 0$ *and* an infinite activity jump component, in practice the models encountered in the financial literature are of two types: either we combine a non-zero diffusion part $\sigma > 0$ with a finite activity jump process (in this case one speaks of a *jump-diffusion* model) or one totally suppresses the diffusion part, in which case frequent small jumps are needed to generate realistic trajectories: these are *infinite activity pure jump models.* Different exponential Lévy models proposed in the financial modelling literature simply correspond to different choices for the Lévy measure ν :

- Compound Poisson jumps: $\sigma > 0$ and ν is a finite measure.
  - Merton model [28]: Gaussian jumps. $\nu = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}}$

  - Superposition of Poisson processes: $\nu = \sum_{k=1}^{n} \lambda_k \delta_{y_k}$ where δ_x is a measure that affects unit mass to point x.
  - Kou model [24] : $\nu(x) = p\alpha_1 e^{-\alpha_1 x} 1_{x>0} + (1-p)\alpha_2 e^{\alpha_2 x} 1_{x<0}$

- Infinite activity models: $\sigma = 0$ and $\int \nu(dx) = \infty$
  - Variance Gamma [26] $\nu(x) = A|x|^{-1} \exp(-\eta_\pm |x|)$
  - Tempered stable[1] processes [23, 10]: $\nu(x) = A_\pm |x|^{-(1+\alpha)} \exp(-\eta_\pm |x|)$
  - Normal inverse gaussian process [4]
  - Hyperbolic and generalized hyperbolic processes [17, 18]
  - Meixner process [36]: $\nu(x) = \frac{Ae^{-ax}}{x \sinh(bx)}$

3 INTEGRO-DIFFERENTIAL EQUATIONS FOR OPTION PRICES

The value of an option with terminal payoff H_T is obtained as a discounted conditional expectation under the (risk-neutral) pricing measure $\mathbb{Q}$: $C_t = E[e^{-r(T-t)} H_T | \mathcal{F}_t]$.

For a European call or put, $H_T = H(S_T)$ with $H(S) = (S-K)^+$ or $H(S) = (K-S)^+$. From the Markov property,

$$C_t = C(t, S) \quad = \quad E[e^{-r(T-t)} H(S_T) | S_t = S].$$

Introducing the change of variable $\tau = T - t$, $x = \ln(S/S_0)$, and defining: $h(x) = H(S_0 e^x)$ and $u(\tau, x) = e^{r\tau} C(T-\tau, S_0 e^x)$, then

$$u(\tau, x) = E[h(x + Y_\tau)]. \tag{6}$$

If h is in the domain of the generator L given by (5), then from (2) we see that u is the solution of the Cauchy problem:

$$\frac{\partial u}{\partial \tau} = Lu, \quad \text{on } (0, T] \times \mathbb{R}; \qquad u(0, x) = h(x), \quad x \in \mathbb{R}. \tag{7}$$

However in all cases of interest in finance, h is not smooth and does not belong to the domain of L. More generally, by applying the Ito formula to $u(t, X_t)$ between 0 and T one can show [7, 15, 31] that any smooth solution $u \in C^{1,2}$ of (7) has the probabilistic representation (6):

Proposition 1 (Feynman–Kac representation for Lévy processes). Assume $\exists a > 0$ such that $\int_{|x|>1} \exp(a|x|)\nu(dx) < \infty$. If $u \in C^{1,2}$ is a classical solution of (7) and its derivatives are bounded by a polynomial function of x, uniformly in $t \in [0, T]$, then u has the probabilistic representation (6).

[1] Also called "truncated Lévy flights" in the physics literature [23, 10] or CGMY processes [8] in the finance literature.

The conditions on u and ν ensure that $u(t, X_t)$ can be represented as a martingale plus a finite variation process; they can be weakened in various ways, see [7, 15, 33].

Similarly, barrier options can be represented in terms of solutions to initial-boundary value problems. Consider for instance an up-and-out call option with maturity T, strike K, and (upper) barrier $U > S_0$. The terminal payoff is given by $H_T = (S_T - K)^+ 1_{T<\theta}$, where $\theta = \inf\{t \geq 0 \mid S_t \geq U\}$ is the first moment when the barrier is crossed.

Due to the strong Markov property of Lévy processes, it is possible to express the value of the option $C_t = e^{-r(T-t)} E[H_T | \mathcal{F}_t]$ as a deterministic function of time t and current stock value S_t before the barrier is crossed. Namely, for any $(t, S) \in [0, T] \times (0, \infty)$ we can define

$$C_b(t, S) \quad = \quad e^{-r(T-t)} \; E[H(Se^{Y_{T-t}}) 1_{T<\theta_t}],$$

where $H(S) = (S - K)^+$ and $\theta_t = \inf\{s \geq t \mid Se^{Y_{s-t}} \geq U\}$, the first exit time after t. Then, $C_t = C_b(t, \; S_t) 1_{t \leq \theta}$ for all $t \leq T$. As in the European case, by going to the log variables we define

$$u_b(\tau, x) = e^{r\tau} C_b(T - \tau, S_0 e^x). \tag{8}$$

Again, if u_b is smooth the Itô formula can be used to show that u_b is a solution of the following initial-boundary-value problem:

$$\frac{\partial u}{\partial \tau} = Lu, \qquad \text{on } (0, T] \times (-\infty, \log(U/S_0)),$$
$$u(0, x) = h(x), \quad x < \log(U/S_0); \qquad u(\tau, x) = 0, \quad x \geq \log(U/S_0).$$

Due to the nonlocal nature of the integral term, "boundary" conditions have to be imposed not only *at* the boundary but outside the boundary.[2] Prices of down-and-out or double barrier options verify similar PIDEs with Dirichlet boundary conditions. The correspondence between the analytic and probabilistic properties discussed above is summarized in Table 1.

4 SMOOTHNESS OF OPTION PRICES

In the case where the log-price X_t has a non-degenerate diffusion component, it is known [7, 20] that the fundamental solution of the pricing PIDE, which correspond to the density of X_t, is in fact a smooth C^∞ function. As a consequence, the option price $u(t, x)$ depends smoothly on the underlying and results such as Proposition 1 allows to use the solution of the PIDE to compute the option price.

In the case of processes with a degenerate diffusion component, such as pure jump models, the smoothness of the conditional expectation as a function of (t, S) does not always hold, as the following example shows.

[2] In fact they extend to any point that the process can jump to from inside the domain.

Table 1: Correspondence between analytical and probabilistic properties

PIDE	**Lévy process**	**Financial interpretation**
x	Starting point of process $Y_t + x$	Log moneyness
Integro-differential operator	Infinitesimal generator	
Initial condition h		Payoff at maturity, expressed in log-price $H(S_0 e^x) = h(x)$
Solution of PIDE $u(t, x)$	$E[h(Y_t + x)]$	Value of option with payoff $H(S_T)$, time to maturity τ
Fundamental solution of Cauchy problem	Transition density $\rho_t(x)$ of Y_t	$e^{-r(T-t)}$ $\times$ Gamma of call option
Zero boundary condition for $x \geq b$	Stopping at first exit from b	Knock-out barrier b
Green function with zero boundary condition for $x \geq b$	Density of stopped process	$e^{-r(T-t)}$ $\times$ Gamma of up-and-out call
Comparison principle	$H \geq 0 \Rightarrow E[H\|\mathcal{F}_t] \geq 0$	Static arbitrage relations

Example 1 (Variance Gamma process). The Variance Gamma process, introduced by Madan & Milne [26], is a pure jump finite variation process with infinite activity, popular in financial modelling. Its Lévy measure has a density given by:

$$\nu(x) = \frac{1}{\kappa|x|} e^{Ax - B|x|} \qquad \text{with} \quad A = \frac{\theta}{\sigma^2} \quad \text{and} \quad B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}. \tag{9}$$

The characteristic function of X_t, the Fourier transform of its distribution, is given by:

$$\Phi_t(u) \quad = \quad (1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u)^{-\frac{t}{\kappa}} \tag{10}$$

$\Phi_t(.)$ decays as $|u|^{-2t/\kappa}$ when $|u| \to \infty$: the decay exponent increases with t. The fundamental solution $\rho(t, x)$ of the PIDE therefore has a degree of regularity which increases *gradually* with t: for $t \in (p\kappa/2, (p+1)\kappa/2)$, the fundamental solution $\rho(t, .)$ is in $C^{p-1}(\mathbb{R})$ but not $C^p(\mathbb{R})$. For $t < \kappa/2$, $\rho(t, .)$ is not even locally bounded. As a consequence, the value of a European binary option defined by the payoff $h(x) = 1_{x \geq x_0}$ is continuous but not differentiable for $t < \kappa/2$.

The case of barrier options is even less regular. As the following example illustrates, if no restriction is imposed on the Lévy process, the value of a barrier option – which is formally the solution of the Dirichlet problem with zero boundary conditions – can even turn out to be discontinuous at all times:

Example 2. Consider $X_t = N_t^1(\lambda_1) - N_t^2(\lambda_2)$ where N_t^i are independent Poisson processes with jump intensities λ_1 and λ_2. Let, for simplicity, $r = 0$. If $\lambda_2 = \lambda_1 e$ then the corresponding price process $S_t = S_0 e^{X_t}$ is a martingale.

Consider now a knock-out option which pays 1 at time T if S_t has not crossed the barrier $U > S_0$ before T, and 0 otherwise:

$$H_T = 1_{T < \theta(S_0)},$$

where $\theta(S) = \inf\{t \geq 0 \mid Se^{X_t} \geq U\}$ is the first exit time if the process starts from S. Let us show that the initial option value

$$C(0, S) = \mathbb{E}[H_T | S_0 = S] = \mathbb{E}[1_{T < \theta(S)}]$$

is not continuous at $S^* = U/e$.

Let $0 < \varepsilon < U - S^*$. By definition, $\theta(S^* + \varepsilon) \leq \theta(S^* - \varepsilon)$. Therefore,

$$\begin{aligned} C(0, S^* - \varepsilon) - C(0, S^* + \varepsilon) &= \mathbb{E}[1_{\{\theta(S^*+\varepsilon) \leq T < \theta(S^* - \varepsilon)\}}] \\ &= \mathbb{Q}\left(\theta(S^* + \varepsilon) \leq T < \theta(S^* - \varepsilon)\right). \end{aligned}$$

Consider the following possibility: $N_T^1 = 1$ et $N_T^2 = 0$, that is, there was one positive and no negative jumps. In this case, if S_t starts from $S^* - \varepsilon$ it stays below U, while starting from $S^* + \varepsilon$ it crosses the barrier. This means that $\theta(S^* + \varepsilon) \le T < \theta(S^* - \varepsilon)$. So,

$$C(0, S^* - \varepsilon) - C(0, S^* + \varepsilon) \;\ge\; \mathbb{Q}(N_T^1 = 1 \;\&\; N_T^2 = 0) \;=\; e^{-\lambda_1 T(e+1)}\lambda_1 T > 0.$$

Thus $S \mapsto C(0, S)$ is discontinuous at $S = S^*$.

This example is a finite activity process without diffusion component. As noted above, this case is not the interesting one in financial modelling. The following result shows that in fact, in most cases of interest, the option price is a continuous function of the underlying :

Proposition 2 (Continuity of European options). If H satisfies the Lipschitz condition (17) then the forward value of a European option defined by $f(\tau, x) = \mathbb{E}[H(S_0 e^{x + r\tau + X_\tau})]$ is continuous on $[0, T] \times \mathbb{R}$.

For a proof, see [15]. Denote by by $C_p^+([0, T] \times \mathbb{R})$ the set of measurable functions on $[0, T] \times \mathbb{R}$ with polynomial growth of degree p at $+\infty$ and bounded on $[0, T] \times \mathbb{R}^-$:

$$\varphi \in C_p^+([0, T] \times \mathbb{R}) \iff \exists C > 0,\ |\varphi(t, x)| \le C(1 + |x|^p\, 1_{x>0}). \tag{11}$$

The pricing function can be shown to have polynomial growth at infinity if the payoff has this property:

Proposition 3 (Polynomial growth). If $H : (0, \infty) \to [0, \infty)$ is Lipschitz: $|H(S_1) - H(S_2)| \le C|S_1 - S_2|$, and there exists $p > 0$, such that:

$$H(S_0 e^x) \le C_1(1 + |x|^p), \tag{12}$$

then $f(\tau, x) = \mathbb{E}[H(S_0 e^{x + r\tau + X_\tau})]$ belongs to $C_p^+([0, T] \times \mathbb{R})$.

In general, one cannot hope for more than Lipschitz continuity; in particular uniform bounds on derivatives, such as the ones required in [31], do not hold in cases of interest in finance where the payoff function H is not smooth, as for call or put options. In these cases, verification theorems such as the Proposition 1 do not apply and the option pricing function should be seen as a *viscosity* solution of the PIDE (7).

5 OPTION PRICES AS VISCOSITY SOLUTIONS

Existence and uniqueness of (classical) solutions for the PIDEs considered above in Sobolev / Hölder spaces have been studied in [7, 20] in the case where the diffusion component is non-degenerate: for a Lévy process this simply means $\sigma > 0$ but more generally these results apply to Markov processes with jumps where the diffusion coefficient is bounded away from zero. However many of the models in the financial modelling literature are pure jump models with $\sigma = 0$, for which such results are not available. In fact, in pure jump models with finite variation Equation (7) is formally of *first* order in the price variable so the effect of the jump term is more like a convection term rather than a diffusion term. A notion of solution which yields existence and uniqueness for such equations without requiring non-degeneracy of coefficients or a priori knowledge of smoothness of solutions is the notion of viscosity solution, introduced by Crandall & Lions for PDEs [16] and extended to integro-differential equations of the type considered here in [1, 3, 32, 35, 37].[3]

Denote by USC (respectively LSC) the class of upper semicontinuous (respectively lower semicontinuous) functions $u : [0,T) \times \mathbb{R} \to \mathbb{R}$. Let $O = (l,u) \subseteq \mathbb{R}$ be an open interval, $\partial O = \{l,u\}$ its boundary, and $g \in C_p^+([0,T] \times \ \mathbb{R} \setminus O)$ a continuous function. Consider the following initial-boundary value problem on $[0,T] \times \mathbb{R}$:

$$\frac{\partial u}{\partial \tau} = Lu, \qquad \text{on } (0,T] \times O, \tag{13}$$

$$u(0,x) = h(x), \quad x \in O; \qquad u(\tau,x) = g(\tau,x), \quad x \notin O. \tag{14}$$

Definition 1 (Viscosity solution). A function $u \in USC$ is a *viscosity subsolution* of (13)–(14) if for any test function $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ and any global maximum point $(\tau,x) \in [0,T] \times \mathbb{R}$ of $u - \varphi$, the following properties are verified:

$$\begin{aligned}
\text{if } (\tau,x) \in (0,T] \times O, &\quad \left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x) \le 0, \qquad (15)\\
\text{if } \tau = 0,\ x \in \overline{O}, &\quad \min\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - h(x)\} \le 0,\\
\text{if } \tau \in (0,T],\ x \in \partial O, &\quad \min\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - g(\tau,x)\} \le 0,\\
\text{if } x \notin \overline{O}, &\quad u(\tau,x) \le g(\tau,x). \qquad (16)
\end{aligned}$$

A function $u \in LSC$ is a *viscosity supersolution* of (13)–(14) if for any test function $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ and any global minimum point $(\tau,x) \in [0,T] \times \mathbb{R}$ of $u - \varphi$, we have:

$$\begin{aligned}
\text{if } (\tau,x) \in (0,T] \times O, &\quad \left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x) \ge 0,\\
\text{if } \tau = 0,\ x \in \overline{O}, &\quad \max\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - h(x)\} \ge 0,\\
\text{if } \tau \in (0,T],\ x \in \partial O, &\quad \max\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - g(\tau,x)\} \ge 0,\\
\text{if } x \notin \overline{O}, &\quad u(\tau,x) \ge g(\tau,x).
\end{aligned}$$

[3] Definitions of viscosity solutions in these papers vary in the choice of test functions; we present here a version which is suitable for option pricing applications.

A function $u \in C_p^+([0,T] \times \mathbb{R})$ is called a *viscosity solution* of (13)–(14) if it is both a subsolution and a supersolution. This function is then continuous on $(0,T] \times \mathbb{R}$.

Note that the initial and boundary conditions are verified in a viscosity sense. The definition also includes the case of initial value problems: $O = \mathbb{R}$. Existence and uniqueness of viscosity solutions for such parabolic integro-differential equations are discussed in [1] in the case where ν is a finite measure and in [3] and [32] for general Lévy measures. Growth conditions other than $u \in C_p^+$ can be considered (see e.g. [1, 3]) with additional conditions on the Lévy measure ν. The main tool for showing uniqueness is the comparison principle: if u, v are viscosity solutions and $u(0,x) \geq v(0,x)$ then $\forall \tau \in [0,T],\ u(\tau,x) \geq v(\tau,x)$. This property can be extended to subsolutions and supersolutions in the following sense:

Proposition 4 (Comparison principle for semi-continuous solutions[1, 22]). If $u \in USC$ is a subsolution and $v \in LSC$ is a supersolution of (13)–(14) then $u \leq v$ on $(0,T] \times \mathbb{R}$.

Proofs and extensions can be found in [1] for the case where ν is a bounded measure; the case of a general Lévy measure has been recently treated in [22].

The following result, whose proof is given in [15] shows that, under rather general conditions on the Lévy triplet and the payoff function, values of European and barrier options can be expressed in terms of (viscosity) solutions of (13)–(14):

Proposition 5 (Option prices as viscosity solutions). Let the payoff function H verify the Lipschitz condition on its domain of definition:

$$|H(S_1) - H(S_2)| \leq C|S_1 - S_2|, \qquad \forall S_1, S_2 \in (S_0 e^l, S_0 e^u) \tag{17}$$

and let $h(x) = H(S_0 e^x)$ have polynomial growth at infinity. Then:

- The forward value of a European option $u(\tau, x)$ defined by (6) is the unique viscosity solution of the Cauchy problem (7) (that is (13)–(14) with $O = \mathbb{R}$).
- Let $u_b(\tau, x)$ be the forward value of a knockout (single or double) barrier option defined by (8). If $u_b(\tau, x)$ is continuous then it is the unique viscosity solution of (13)–(14) (with $g \equiv 0$).

The hypotheses above on the payoff function apply to put options, single-barrier knockout puts, double barrier knockout options and also to the log-contract. One can then retrieve call options by put-call parity. For barrier options with rebate, the zero boundary condition has to be replaced by the value of the rebate, as in the case of diffusion models. A discussion of sufficient conditions for continuity of value functions for barrier options is given in [15].

A popular method for pricing European options in exp-Lévy models is the Fourier method proposed by Carr & Madan [9], which is the method of choice when analytic expressions are available for the characteristic function of the Lévy process X_t. However this method does not extend to barrier options or American options. Numerical solution of PIDEs allows efficient pricing of European and barrier options on assets with jumps and does not require analytic formulae for characteristic functions. Numerical methods for PIDEs are discussed in [14, 27]. In particular the notion of viscosity solution turns out to be convenient for analyzing the convergence of finite difference schemes, without requiring smoothness with respect to the underlying [14].

6 MODEL CALIBRATION: AN ILL POSED INVERSE PROBLEM AND ITS REGULARIZATION

A preliminary step in using an option pricing model is to obtain model parameters – here the characteristic triplet of the Lévy process – from market data by calibrating the model to market prices of (liquid) call options. The calibration problem for exp-Lévy models consists of identifying the Lévy measure ν and the volatility σ from a set of observations of call option prices:

Calibration Problem 1. *Given the market prices of call options $C_0^*(T_i, K_i), i = 1..N$ at $t = 0$, construct a Lévy process $(X_t)_{t\geq 0}$ such that the discounted asset price $S_t e^{-rt} = \exp X_t$ is a martingale and the market call option prices $C_0^*(T_i, K_i)$ coincide with the prices of these options computed in the exponential Lévy model driven by X:*

$$\forall i \in \{1, .., N\},\ C_0^*(T_i, K_i) = e^{-rT_i} E[(S_{T_i} - K_i)^+ | S_0] = e^{-rT_i} E[(S_0 e^{rT_i + X_{T_i}} - K_i)^+]. \quad (18)$$

Problem 1 can be seen as a generalized moment problem for the Lévy process X, which is typically an ill posed problem when the observations are finite and/or noisy: there may be no solution at all or an infinite number of solutions and the dependence of the solution(s) on option prices may be discontinuous, which results in numerical instabilities in the calibration algorithm.

In this section, we first describe the commonly used nonlinear least squares method for solving this problem and show its shortcomings. Next, we introduce a regularization approach using relative entropy, discuss its properties and its numerical implementation. This section is based on [12, 13].

6.1 Nonlinear least squares

The calibration problem 1 may have no solution, either because the model is misspecified or because the observed option prices, defined up to a bid-ask spread, do not lie within the model range. An approach used to in many empirical studies of option pricing models [2, 6] is to minimize the (squared) difference between model prices $C^{\sigma,\nu}$ and market prices C^*, summed over all liquid options available in the market:

$$(\sigma^*, \nu^*) = \arg\inf_{\sigma,\nu} \epsilon(\sigma, \nu) \qquad \epsilon(\sigma, \nu) = \sum_{i=1}^{N} w_i |C^{\sigma,\nu}(t = 0, S_0, T_i, K_i) - C_0^*(T_i, K_i)|^2 \quad (19)$$

where w_i are positive weights chosen to balance the magnitude of the different terms; a typical choice is to take w_i^{-1} as the Black–Scholes Vega of the call option (T_i, K_i). Gradient–based methods are then used to perform the numerical minimization in (19). Unfortunately this is a nonlinear, nonconvex optimization problem where gradient methods may give erroneous results. Figure 1 shows the shape of the nonlinear least squares criterion in the case of two popular models, the Merton model (left) and the Variance Gamma model (right), computed for DAX index options. In the Merton model we observe a continuum of minima corresponding to the difficulty of distinguishing, using the option prices, the effect of jumps and volatility. In the Variance Gamma case we observe two quite distinct parameter sets giving similar calibration performance. At a theoretical

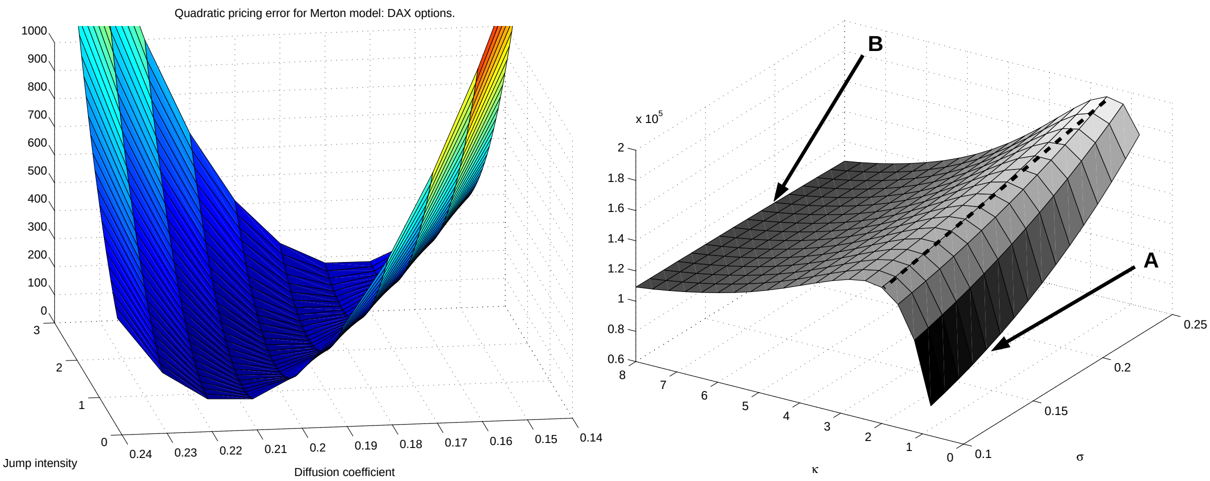

Figure 1: Sum of squared differences between market prices (DAX options maturing in 10 weeks) and model prices in the Merton model (left) and the variance gamma model as a function of σ and κ, the third parameter being fixed.

level, while conditions can be derived [13] for well-posedness of solutions for the nonlinear least square problem (19) they turn out to be rather restrictive and imply that the range of model parameters is already quite well known a priori.

6.2 Regularization by relative entropy

One way to enforce uniqueness and stability of the solution is to inject prior information into the problem by specifying a prior (exp-Lévy) model $\mathbb{Q}_0$ and add to the least-squares criterion (22) a convex *penalization* term $F(\sigma, \nu)$ which measures the deviation of the exp-Lévy model defined by (σ, ν) from the prior model $\mathbb{Q}_0$:

$$(\sigma^*, \nu^*) = \arg \inf_{\sigma, \nu} \sum_{i=1}^{N} w_i |C^{\sigma,\nu}(t=0, S_0, T_i, K_i) - C_0^*(T_i, K_i)|^2 + \alpha F(\sigma, \nu) \tag{20}$$

Problem (20) can be understood as that of finding an exponential Lévy model satisfying the conditions (1), which is closest in some sense – defined by F – to a prior exp-Lévy model. The convex term αF is called a *regularization* term and is used to make the problem well-posed. The regularization parameter $\alpha > 0$ is chosen to ensure a tradeoff between precision and stability [19].

A common choice for the regularization term is the *relative entropy* or *Kullback Leibler distance* $\mathcal{E}(\mathbb{Q}, \mathbb{Q}_0)$ of the the pricing measure $\mathbb{Q}$ with respect to the prior model $\mathbb{Q}_0$:

$$F(\sigma, \nu) = \mathcal{E}(\mathbb{Q}, \mathbb{Q}_0) = E^{\mathbb{Q}}\left[\ln \frac{d\mathbb{Q}}{d\mathbb{Q}_0}\right]$$

In addition to being convex, relative entropy acts also as a barrier function for the positivity and absolute continuity constraints on (σ, ν). In the case of (risk-neutral) exp-Lévy models, the relative entropy is easily computable in terms of the volatility σ and calibrated Lévy measure ν [21, Theorem IV.4.39]:

$$\mathcal{E}(\mathbb{Q}, \mathbb{Q}_0) = H(\nu) = \frac{T}{2\sigma^2}\left\{\int_{-\infty}^{\infty}(e^x - 1)(\nu - \nu_0)(dx)\right\}^2 + T\int_{-\infty}^{\infty}\left(\frac{d\nu}{d\nu_0}\ln\frac{d\nu}{d\nu_0} + 1 - \frac{d\nu}{d\nu_0}\right)\nu_0(dx) \quad (21)$$

$H(\nu)$ is a convex functional of the Lévy measure ν, with a unique minimum at $\nu = \nu_0$. Relative entropy plays the role of a pseudo-distance of the (risk-neutral) measure from the prior and minimizing it corresponds to adding the least possible amount of information to the prior in order to correctly reproduce observed option prices . The explicit expression (21) of relative entropy in terms of the Lévy measure allows to construct an efficient numerical method for finding the minimal entropy Lévy process, compatible with a set of observed option prices.

The calibration problem now takes the form:

Calibration Problem 2. *Given a prior exponential Lévy model $\mathbb{Q}_0$ with characteristics (σ_0, ν_0) find a Lévy measure ν which minimizes*

$$\mathcal{J}(\nu) = \alpha H(\nu) + \sum_{i=1}^{N} w_i(C_0^{\nu}(T_i, K_i) - C_0^*(T_i, K_i))^2 \quad (22)$$

where $H(\nu)$ is the relative entropy of the risk neutral measure with respect to the prior, whose expression is given by (21).

The functional (22) consists of two parts: the relative entropy functional, which is convex in its argument ν and the quadratic pricing error which measures the precision of calibration. The coefficient α, called *regularization parameter* defines the relative importance of the two terms: it characterizes the trade-off between prior knowledge of the Lévy measure and the information contained in option prices.

Regularization using relative entropy allows to obtain existence of solutions and continuity with respect to market data (input option prices) [13]:

Proposition 6 (Regularized calibration problem). Let the prior model $\mathbb{Q}_0$ be an exp-Lévy process with jumps bounded from above.

- Existence: for each data set $C^* = (C^*(T_i, K_i), i = 1..N)$ there exists an exponential Lévy model $\mathbb{Q}$ which is a solution of the regularized calibration problem (22).
- The solution(s) depend continuously on the market prices: Let $\{C^n\}_{n\geq 1}$ and C^* be data sets of option prices such that
$$\|C^n - C^*\| \underset{n\to\infty}{\to} 0.$$
For each $n \geq 1$, let $\mathbb{Q}_n$ be a solution of the calibration problem (22) with data C^n and prior $\mathbb{Q}_0$. Then $(\mathbb{Q}_n)_{n\geq 1}$ has a weakly converging subsequence and the limit of every convergent subsequence of $(\mathbb{Q}_n)_{n\geq 1}$ is a solution of calibration problem (22) with data C^* and prior $\mathbb{Q}_0$.
- Stability with respect to the prior: Let $\mathbb{P}^n$ be a sequence of probability measures corresponding to exp-Lévy models with jumps uniformly bounded above by some constant $B > 0$, weakly converging to $\mathbb{Q}_0$. Let $\mathbb{Q}^n$ be solution of (22) with prior $\mathbb{P}^n$. Then $(\mathbb{Q}^n)_{n\geq 1}$ has a weakly converging subsequence and the limit of every weakly converging subsequence is a solution of (22) with prior $\mathbb{Q}_0$.

6.3 Numerical implementation

An important feature of the objective functional (22) is that its (directional) derivatives can be computed explicitly [12]. This allows to use a gradient-based method to solve the regularized optimization problem (22). The regularization parameter α in (22) is determined using the Morozov discrepancy principle, as follows. First, we remark that the observed option prices $C^* = (C^*(K_i, T_i), i = 1..N)$ are defined up to a bid–ask spread; this allows to define an a priori level for the quadratic pricing error:

$$\epsilon_0 = \sum_{i=1}^{N} w_i |C_i^{bid} - C_i^{ask}|^2 \tag{23}$$

Now let (σ, ν_α) be the solution of (22) for a given regularization parameter $\alpha > 0$. Then the *a posteriori* quadratic pricing error is given by $\epsilon(\sigma, \nu_\alpha)$, which one expects to be a bit larger than ϵ_0 since by adding the entropy term we have sacrificed some precision in order to gain in stability. The Morozov discrepancy principle [29] consists in authorizing a loss of precision that is of the same order as the a priori error by choosing α such that

$$\epsilon_0 \simeq \epsilon(\sigma, \nu_\alpha) \tag{24}$$

This equation need not be solved precisely: one needs simply to obtain the correct order of magnitude for α, which is then substituted in (22) and solved to obtain the solution of the regularized problem. Figure 2 illustrates the performance of the algorithm on a

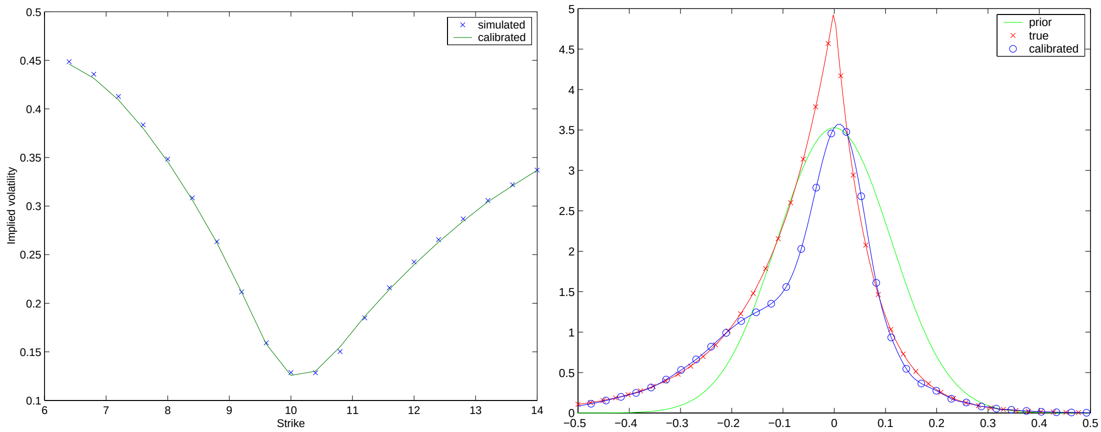

Figure 2: Left: calibrated vs simulated (true) implied volatilities for Kou model [24]. Right: double exponential Lévy measure used to generate data set, prior (Gaussian measure) and calibrated measure.

simulated data set: a set of 20 option prices simulated from a Kou jump–diffusion model [24] with diffusion coefficient $\sigma = 0.1$ was used and the prior model $\mathbb{Q}_0$ was chosen to be a Merton model [28] with a (biased) diffusion coefficient $\sigma = 0.105$ and a Gaussian measure for ν_0. The left figure shows the calibration performance: implied volatilities generated from the Kou model are retrieved to within a few basis points. The right figure shows the reconstructed Lévy measure: the positive bias in the diffusion coefficient of the prior is compensated by a decrease in the intensity of small jumps. This examples illustrates several points. First, it is difficult to distinguish the effect of (small) jumps from volatility in prices of European options. Second, a small number of options – twenty in this example – is sufficient to retrieve the Lévy measure. Third, a bias/error in the estimation of volatility is compensated for by an opposite bias in the intensity of small jumps, resulting in a precise fit of option prices.

7 PERSPECTIVES

The incorporation of jumps into option pricing models has led to models a more realistic vision of financial risk. The use of these models, initially hampered by the lack of suitable computational methods, has been substantially eased in recent years by the availability of efficient numerical methods: for exponential Lévy models, efficient numerical methods have been developed for pricing European and exotic options and calibration of model parameters.

A richer class of models, interesting for applications, is the class of stochastic volatility models with jumps.[4] While Fourier–based methods can still be applied for pricing European options in such models, efficient numerical methods for calibration and pricing of exotic options remain to be developed.

Another direction where many modelling and computational issues remain is multi-asset models with jumps. While the computational finance literature has primarily focused on one–dimensional problems, most applications concern multidimensional ones: multi-asset options, interest rate options and portfolio optimization. We hope that some readers will become sufficiently interested to delve into these subjects!

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