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A Stochastic Partial Differential Equation Model for Limit Order Book Dynamics*

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Abstract. We propose an analytically tractable class of models for the dynamics of a limit order book, described through a stochastic partial differential equation with multiplicative noise for the order book centered at the mid-price, along with stochastic dynamics for the mid-price which is consistent with the order flow dynamics. We provide conditions under which the model admits a finite-dimensional realization driven by a (low-dimensional) Markov process, leading to efficient methods for estimation and computation. We study two examples of parsimonious models in this class: a two-factor model and a model in which the order book depth is mean reverting. For each model we perform a detailed analysis of the role of different parameters, study the dynamics of the price, order book depth, volume, and order imbalance, provide an intuitive financial interpretation of the variables involved, and show how the model reproduces statistical properties of price changes, market depth, and order flow in limit order markets.

Key words. limit order book, market microstructure, stochastic PDE, moving boundary problem, volatility, liquidity

AMS subject classifications. 35R60, 60H15, 91B26, 91G80

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Financial instruments such as stocks and futures are increasingly traded in electronic, order-driven markets, in which orders to buy and sell are centralized in a *limit order book* and market orders are executed against the best available offers in the limit order book. The dynamics of prices in such markets are interesting not only from the viewpoint of market participants—for trading and order execution—but also from a fundamental perspective, since they provide a detailed view of the dynamics of supply and demand and their role in price formation.

The availability of a large amount of high frequency data on order flow, transactions, and price dynamics on these markets has instigated a line of research which, in contrast to traditional market microstructure models which make assumptions on the behavior and preferences of various types of agents, focuses on the *statistical modeling* of aggregate order flow and its relation with price dynamics, in a quest to understand the interplay between price dynamics and order flow of various market participants (Cont, 2011).

A fruitful line of approach to these questions has been to model the stochastic dynamics of the *limit order book*, which centralizes all buy and sell orders, either as a queueing system

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(Luckock, 2003; Smith et al., 2003; Cont, Stoikov, and Talreja, 2010; Cont and de Larrard, 2012; Cont and de Larrard, 2013; Kelly and Yudovina, 2018) or, at a coarse-grained level, through a (stochastic) PDE describing the evolution of the distribution of buy and sell orders (Lasry and Lions, 2007; Caffarelli, Markowich, and Pietschmann, 2011; Burger et al., 2013; Carmona and Webster, 2019; Markowich, Teichmann, and Wolfram, 2016; Hambly, Kalsi, and Newbury, 2020; Horst and Kreher, 2018). These PDE models may be viewed as scaling limits of discrete point process models (Cont and de Larrard, 2012; Hambly, Kalsi, and Newbury, 2020; Horst and Kreher, 2018).

Although joint modeling of order flow at all price levels in the limit order book is more appealling, (stochastic) PDE models have lacked the analytical and computational tractability needed for applications; as a result, most analytical results have been derived using reduced-form models of the best bid-ask queues (Cont and de Larrard, 2012; Cont and de Larrard, 2013; Chavez-Casillas and Figueroa-Lopez, 2017; Huang, Lehalle, and Rosenbaum, 2017).

We propose a class of stochastic models for the dynamics of the limit order book which represent the dynamics of the entire order book while retaining at the same time the analytical and computational tractability of low-dimensional Markovian models and that provides realistic dynamics for the joint dynamics of the market price and order book depth. Starting with a description of the dynamics of the limit order book via a stochastic partial differential equation (SPDE) with multiplicative noise, we show that in many cases, the solutions of this equation may be parameterized in terms of a low-dimensional diffusion process, which then makes the model computationally tractable. In particular, we are able to derive analytical relations between model parameters and various observable quantities. This feature may be used for calibrating model parameters to match statistical features of the order flow and leads to empirically testable predictions, which we proceed to test using high-frequency time series of order flow in electronic equity markets.

Outline. Section 1 introduces a description of the dynamics of a limit order book through an SPDE. We describe the various terms in the equation and their interpretation and discuss the implications for price dynamics (section 1.3). This class of models is part of a more general family of SPDEs driven by semimartingales, introduced in section 1.5 and studied in section 2.

We then focus on two analytically tractable examples: a two-factor model (section 3) and a model with mean-reverting depth and imbalance (section 4). For each model we perform a detailed analysis of the role of different parameters and study the dynamics of the price, order book depth, volume, and order imbalance, provide an intuitive financial interpretation of the variables involved, and show how the model may be estimated from financial time series of price, volume, and order flow.

1. An SPDE model for limit order book dynamics.

We consider a market for a financial asset (stock, futures contract, etc.) in which buyers and sellers may submit *limit orders* to buy or sell a certain quantity of the asset at a certain price, and *market orders* for immediate execution against the best available price.[1] Limit orders awaiting execution are collected in the *limit order book*, an example of which is shown in Figure 1: at any time t, the state of

[1]In the following we do not distinguish market orders and marketable limit orders, i.e., limit orders with a price better than the best price on the opposite side.

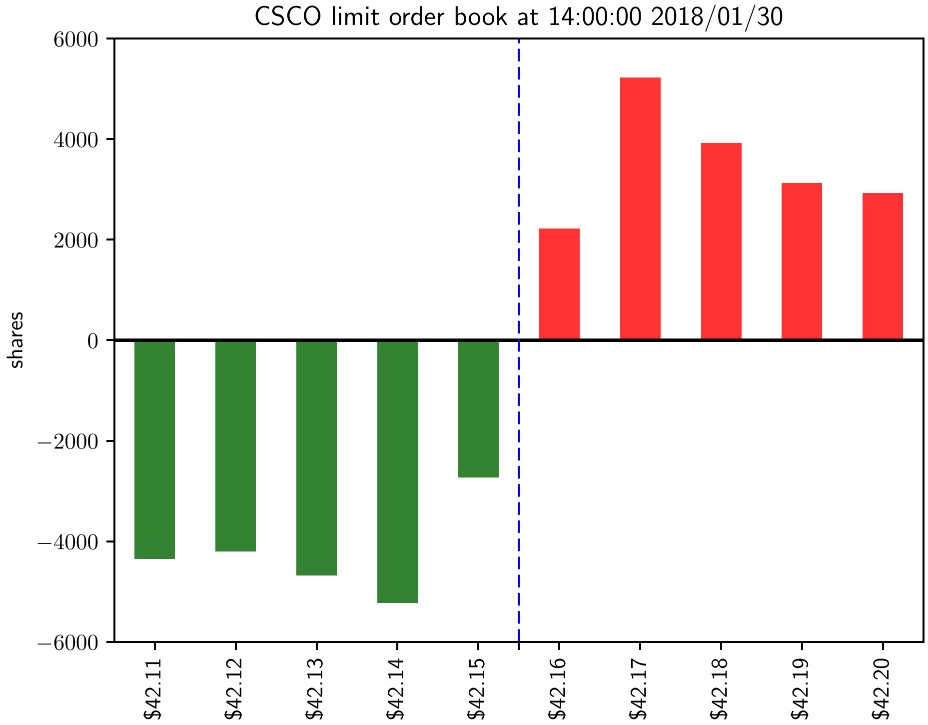

Figure 1. *Snapshot of the NASDAQ limit order book for CISCO shares (January 30, 2018), displaying outstanding buy orders (green) and sell orders (red) awaiting execution at different prices. The highest buying price (\$42.15 in this example) is the bid and the lowest selling price (\$42.16) is the ask.*

the limit order book is summarized by the volume $V(t,p)$ of orders awaiting execution at price levels p on a grid with mesh size given by the minimum price increment or *tick* size δ. By convention we associate negative volumes with buy orders and positive volumes with sell orders, as shown in Figure 1. An admissible order book configuration is then represented by a function $p \mapsto V(p)$ such that

$$0 < s^b(V) := \sup\{p > 0, \quad V(p) < 0\} \leq s^a(V) := \inf\{p > 0, \quad V(p) > 0\} < \infty.$$

$s^b(V)$ (resp., $s^a(V)$) is called the bid (resp., ask) price and represents the price associated with the best buy (resp., sell) offer. The quantity

$$S = \frac{s^a(V) + s^b(V)}{2}$$

is called the *mid-price* and the difference $s^a(V) - s^b(V)$ is called the bid-ask spread. In the example shown in Figure 1, $s^b(V) = 42.15, s^a(V) = 42.16$ and the bid-ask spread is equal in this case to the tick size, which is 1 cent.

One modeling approach has been to represent the dynamics of $V(t,p)$ as a spatial (marked) point process (Luckock, 2003; Cont, Stoikov, and Talreja, 2010; Cont and de Larrard, 2013; Kelly and Yudovina, 2018). These models preserve the discrete nature of the dynamics at high frequencies but can become computationally challenging as one tries to incorporate realistic dynamics. In particular, price dynamics, which is endogenous in such models, is difficult to study, even when the order flow is a Poisson point process.

When the bid-ask spread and tick size δ are much smaller than the price level, as is often the case, another modeling approach is to use a continuum approximation for the order book, describing it through its *density* $v(t,p)$ representing the volume of orders per unit price:

$$V(t,p) \simeq v(t,p)\delta.$$

The evolution of the density of buy and sell orders is then described through a PDE. A deterministic description of the dynamics of order densities through a system of coupled PDEs was proposed by Lasry and Lions (2007) and studied in detail by Chayes et al. (2009), Caffarelli, Markowich, and Pietschmann (2011), and Burger et al. (2013). In the Lasry–Lions model, the evolution of the density of buy and sell orders is described by a pair of diffusion equations coupled through the dynamics of the price, which represents the free boundary between prices of buy and sell orders. This model is appealing in many respects, especially in terms of analytical tractability, but leads to a deterministic price process which decays to a constant price, so does not provide any insight into the relation between liquidity, depth, order flow, and price volatility. Markowich, Teichmann, and Wolfram (2016) explore some stochastic extensions of this model but essentially show that these extensions do not provide realistic price dynamics.

We adopt here this continuum approach for the description of the limit order book but describe instead its dynamics through a *stochastic* PDE, paying close attention to price dynamics and its relation with order flow.

The model we propose shares some features with Lasry and Lions (2007) but also has some essential differences. Unlike the Lasry–Lions model, which is a free boundary problem in which the dynamics of the price is implicitly determined, we formulate the model as an SPDE in relative price coordinates, which leads to a *stochastic* moving boundary problem in absolute price coordinates. This leads to a more realistic joint dynamics for the market price and order book depth which can be related to empirical observations. Our model also relates to the classes of models studied in Horst and Kreher (2018) and Hambly, Kalsi, and Newbury (2020) as scaling limits of discrete queueing systems.

We now describe our model in some detail.

1.1. State variables and scaling transformations.

We focus on the case where the tick size δ and the bid-ask spread are small compared to the typical price level and consider a limit order book described in terms of a mid-price S_t and the density $v(t,p)$ of orders at each price level p, representing buy orders for $p < S_t$ and sell orders for $p > S_t$. We use the convention, shown in Figure 1, of representing buy orders with a negative sign and sell orders with a positive sign, so

$$v(t,p) \le 0 \quad \text{for} \quad p < S_t \quad \text{and} \quad v(t,p) \ge 0 \quad \text{for} \quad p > S_t.$$

Limit orders are executed against market orders according to price priority and their position in the queue; execution of a limit order occurs only if they are located at the best (buy/sell) prices. This means that price dynamics is determined by the interaction of market orders with limit orders of opposite type at or near the *interface* defined by the best price (Cont, Stoikov, and Talreja, 2010). Due to this fact, most limit orders are submitted close to the best price levels: the frequency of limit order submissions is highly inhomogeneous as a function of distance to the best price and concentrated near the best price. As shown in previous empirical studies, order flow intensity at a given distance from the best price can be considered as a stationary variable in a first approximation (Bouchaud, Farmer, and Lillo, 2009; Cont, Stoikov, and Talreja, 2010). For this reason, in a stochastic description it is more convenient to model the dynamics of order flow in the reference frame of the (mid-)price S_t. We define

$$u_t(x) = v(t, S_t + x),$$

where x represents a distance from the mid-price. We refer to u_t as the *centered* order book density.

The simplest way of centering is to set $x(p) = p - S_t$ but other, nonlinear, scalings may be of interest. Although limit orders may be placed at any distance from the bid/ask prices, price dynamics is dominated by the behavior of the order book a few levels above and below the mid-price (Cont and de Larrard, 2012). This region becomes infinitesimal if the tick size δ is naively scaled to zero, suggesting that the correct scaling limit is instead one in which we choose as coordinate a scaled version $(p - S_t)$, as classically done in boundary layer analysis of PDEs (Schlichting and Gersten, 2017), in order to zoom into the relevant region:

$$x(p) := -(S_t - p)^a, \quad p < S_t, \qquad x(p) = (p - S_t)^a, \quad p > S_t, \qquad a > 0, \tag{1.1}$$

for bid and ask sides, respectively. We will consider examples of such nonlinear scalings when discussing applications to high-frequency data in sections 3 and 4.

These arguments also justify limiting the range of the argument x to a bounded interval $[-L, +L]$, setting $u_t(x) = 0$ for $x \notin (-L, L)$. This amounts to assuming that no orders are submitted at price levels at distances $|x| \geq L$ from the mid-price and that orders previously submitted at some price p are canceled as soon as $|S_t - p| \geq L$, i.e., when the mid-price S_t moves away from p by more than L. When L is a large multiple of daily volatility, this is a realistic assumption. In some markets (for example, futures contracts), limit orders can be in fact submitted only within a range $\pm L$ of the mid-price.

1.2. Dynamics of the centered limit order book. Empirical studies on intraday order flow in electronic markets reveal the coexistence of two very different types of order flow operating at different frequencies (Lehalle and Laruelle, 2018).

On one hand, we observe the submission (and cancellation) of orders queueing at various price levels on both sides of the market price by regular market participants. Cancellation may occur in several ways: we distinguish *outright* cancellations, which we model as proportional to current queue size, from cancellations with replacement ("order modifications"), in which an order is canceled and immediately replaced by another one of the same type, usually at a neighboring price limit. The former results in a net decrease in the volume of the order book whereas the latter is *conservative* and simply shifts orders across neighboring levels of the book. Further decomposing this conservative flow into a symmetric and an antisymmetric part leads to two terms in the dynamics of u_t: a *diffusion* term representing the cancellation of orders and their (symmetric) replacement by orders at neighboring price levels and a *convection* (or transport) term representing the cancellation of orders and their replacement by orders closer to the mid-price. Denoting by ∇ the gradient in the variable x, the net effect of this order flow on the order book may thus be described as a superposition of

- a term $f^b(x)$ (resp., $f^a(x)$) representing the rate of buy (resp., sell) order submissions at a distance x from the best price;
- a term $\alpha_b\, u_t(x)$ (resp., $\alpha_a\, u_t(x)$) representing (outright) proportional cancellation of limit buy (resp., sell) orders at a distance x from the mid-price (where $\alpha_a, \alpha_b \le 0$);
- a convection term $-\beta_b \nabla u_t(x)$ (resp., $+\beta_a \nabla u_t(x)$) with $\beta_a, \beta_b > 0$ which models the replacement of buy (resp., sell) orders by orders closer to the mid-price (i.e., closer to $x = 0$, hence the signs in these terms): in the reference frame where the origin is the mid-price, this translates into a flow of volume toward the origin;
- a diffusion term $\eta_b \Delta u_t(x)$ (resp., $\eta_a \Delta u_t(x)$) which represents the cancellation and symmetric replacement of orders at a distance x from the mid-price.

Another component of order flow is the one generated by high-frequency traders (HFT). These market participants buy and sell at very high frequency and under tight inventory constraints, submitting and canceling large volumes of limit orders near the mid-price and resulting in an order flow whose net contribution to total order book volume is zero on average over longer time intervals but whose sign over small time intervals fluctuates at high frequency. At the coarse-grained time scale of the average (non-high-frequency) market participants, these features may be modeled as a multiplicative noise term of the form

- $\sigma_b u_t(x) dW^b$ for buy orders ($x < 0$) and $\sigma_a u_t(x) dW^a$ for sell orders ($x > 0$),

where (W^a, W^b) is a two-dimensional Wiener process (with possibly correlated components). The multiplicative nature of the noise accounts for the high-frequency cancellations associated with HFT orders.

The impact of these different order flow components may be summarized by the following SPDE for the centered order book density u:

$$
\begin{aligned}
\mathrm{d}u_t(x) &= [\eta_a \Delta u_t(x) + \beta_a \nabla u_t(x) + \alpha_a u_t(x) + f^a(x)]\, \mathrm{d}t + \sigma_a u_t(x)\, \mathrm{d}W_t^a, \quad x \in (0, L),\\
\mathrm{d}u_t(x) &= \left[\eta_b \Delta u_t(x) - \beta_b \nabla u_t(x) + \alpha_b u_t(x) - f^b(x)\right] \mathrm{d}t + \sigma_b u_t(x)\, \mathrm{d}W_t^b, \quad x \in (-L, 0),\\
&\qquad u_t(x) \le 0, \quad x < 0, \qquad u_t(x) \ge 0, \quad x > 0,\\
&\qquad u_t(0+) = u_t(0-) = 0, \qquad u_t(-L) = u_t(L) = 0.
\end{aligned} \tag{1.2}
$$

Here η_a, η_b, β_b, β_a, σ_a, $\sigma_b \in (0, \infty)$, $\alpha_a, \alpha_b \le 0$ and f^a, $f^b \colon I \to [0, \infty)$ although the equation may be equally considered without these sign restrictions.

Note that, unlike the Lasry–Lions model, there is no "smooth pasting" condition at $x = 0$: in general $\nabla u_t(0+) \ne \nabla u_t(0-)$; the difference $\nabla u_t(0+) - \nabla u_t(0-)$ is in fact random and represents an imbalance in the flow of buy and sell orders, which drives price dynamics. This important feature is discussed in section 1.3 below.

Remark 1.1. In simple price impact models used in the literature on optimal trade execution it is assumed that the relation between price impact and order size is deterministic. This corresponds to the case $\alpha u + f = \beta = \sigma = \eta = 0$ which leads to a constant centered order book profile $u_t(.) = u_0(.)$. These terms thus correspond to deformations of the centered order book profile due to new order book events and lead to a stochastic market impact of trades dependent on the current state of the order book.

The existence of a solution satisfying the boundary and sign constraints is not obvious but we will see in section 2 that (1.2) is well-posed: it follows from Da Prato and Zabczyk (2014, Theorem 6.7) and Milian (2002, Theorem 3) that when f_a, $f_b \in L^2(I)$, then for all $u_0 \in L^2(I)$ there exists a unique weak solution of (1.2) (see Definition 2.2 below) and when $u_0|_{(0,L)} \geq 0$ and $u_0|_{(-L,0)} \leq 0$ this solution satisfies

$$u_t|_{(0,L)} \leq 0, \qquad u_t|_{(-L,0)} \geq 0. \tag{1.3}$$

We will study the mathematical properties of the solution in more detail below.

1.3. Price dynamics. The dynamics of the limit order book determines the dynamics of the bid and ask price, which corresponds to the location of the best (buy and sell) orders. The dynamics of the price should thus be related to the arrival and execution of orders in the order book.

To understand the relation between price dynamics and order flow, let us take a step back and consider an order book with discrete price levels, multiples of a tick size δ, D^b orders per level on the bid side, and D^a orders per level on the ask side. Price changes during a time interval $[t, t + \Delta t]$ are triggered through the interaction of the net order flow, or *order flow imbalance* and the outstanding limit orders at the top of the order book (Cont, Kukanov, and Stoikov, 2014). As illustrated in Figure 2, an order flow imbalance of $\Delta D_t^a > 0$ on the ask side over a short time interval $[t, t+\Delta t]$ represents an excess of buy orders, which will then be executed against limit sell orders sitting on the ask side and move the ask price by $\Delta D_t^a / D^a$ ticks, resulting in a price move of $\delta\ \Delta D_t^a / D^a$. Similarly, an order flow imbalance ΔD_t^b on the bid side will move the bid price up by $\Delta D_t^b / D^b$ ticks. Using our sign conventions for buy/sell volumes, this leads to the dynamics

$$\Delta s_t^b = \delta \frac{\Delta D_t^b}{D_t^b} \qquad \Delta s_t^a = -\delta \frac{\Delta D_t^a}{D_t^a},$$

so the dynamics of the mid-price $s_t = (s_t^b + s_t^a)/2$ is given by

$$\Delta S_t = \frac{\delta}{2} \left(\frac{\Delta D_t^b}{D_t^b} - \frac{\Delta D_t^a}{D_t^a} \right). \tag{1.4}$$

This relation is exact (up to rounding) in the case of a discrete order book with constant depth per level (and thus no empty levels), as shown in Figure 2. However, in a dynamic setting where the order book may have an arbitrary profile which randomly shifts at each instant, one can only expect a "homogenized" version of (1.4) to hold:

$$\Delta S_t = \theta \left(\frac{\Delta D_t^b}{D_t^b} - \frac{\Delta D_t^a}{D_t^a} \right), \tag{1.5}$$

where θ is an *impact* coefficient which relates order imbalance to price movements. This relation between order flow imbalance and price movements has been empirically verified in

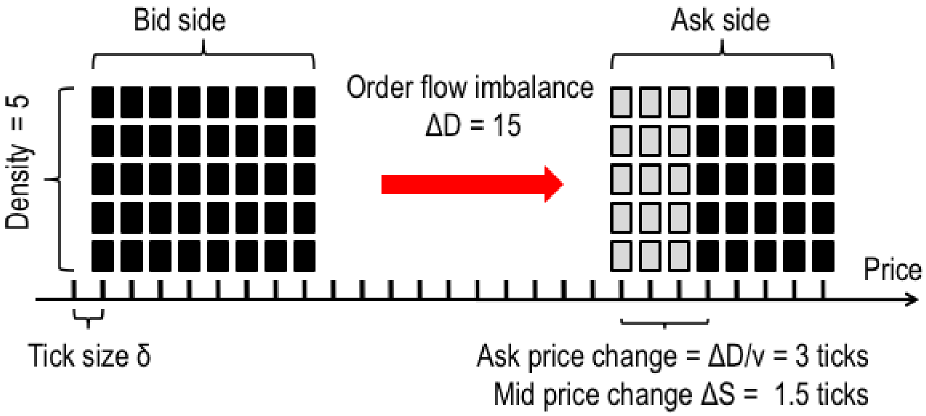

Figure 2. *Impact of order flow imbalance on the order book and the price.*

equity markets (Cont, Kukanov, and Stoikov, 2014), and we shall use it as a basis for defining the relation between price dynamics and order flow in our model.

Let us now see how the relation (1.5) translates in terms of the variables in our model. Denote by D_t^b (resp., D_t^a) the volume of buy (resp., sell) limit orders at the top of the book (i.e., the first or average of the first few levels). Given a mid-price $S \in \mathbb{R}_+$, we define a scaling transformation $x\colon [S, S+L] \to [0,\infty)$ as discussed in section 1.1, with continuously differentiable inverse and such that $x(S) = 0$. The volume D^a in the best ask queue is then given by

$$D^a = \int_s^{s+\delta} u(x(p))\, \mathrm{d}p = \int_0^{x(s+\delta)} u(y)(x^{-1})'(y)\, \mathrm{d}y. \tag{1.6}$$

D^b may be similarly defined for the bid side. These quantities represent the depth at the top of the book; we will refer to them as "market depth." In the case of linear scaling $x(p) = p - S$, using $u(0) = 0$ a second-order expansion in $\delta > 0$ yields

$$D^a = \int_0^{\delta} u(x)\, \mathrm{d}x \approx \delta u(0+) + \frac{\delta^2}{2}\nabla u(0+) = \frac{\delta^2}{2}\nabla u(0+). \tag{1.7}$$

Similarly, for the bid side

$$D^b \approx \frac{\delta^2}{2}\nabla u\,(0-)\,. \tag{1.8}$$

Substituting these expressions in (1.5), we obtain the following dynamics of the mid-price:

$$\mathrm{d}S_t = \theta\left(\frac{\mathrm{d}D_t^b}{D_t^b} - \frac{\mathrm{d}D_t^a}{D_t^a}\right) = \theta\left(\frac{\mathrm{d}\nabla u_t(0-)}{\nabla u_t(0-)} - \frac{\mathrm{d}\nabla u_t(0+)}{\nabla u_t(0+)}\right). \tag{1.9}$$

We observe that price dynamics is entirely determined by the order flow at the top of the book and the depth of the limit order book around the mid-price. The tick size δ, used in the derivation, does not appear anymore in (1.9). The only trace of the microstructure is the impact coefficient θ which relates the order flow imbalance to the magnitude of the price change, and whose amplitude may vary across assets.

Remark 1.2. Equation (1.9) requires left and right differentiability of u at the origin. This can be guaranteed whenever u_t takes values in the Sobolev space $H^{2\gamma}(I)$ for some $\gamma > 3/4$, which will be the case in our model. Note, however, that, in contrast to Lasry and Lions (2007), in general $\nabla u(0+) \neq \nabla u(0-)$: the difference between these two quantities is proportional to the order flow imbalance which drives price moves.

Remark 1.3. As noted in Remark 1.1, the case $\alpha u + f = \beta = \sigma = \eta = 0$ corresponds to a constant centered order book profile $u_t = u_0$. In this case, (1.9) implies $dS_t = 0$, i.e., the price is constant, which is consistent with a zero net order flow. This is a (desirable) consequence of the consistency between the price dynamics (1.9) and the order book dynamics (1.2).

1.4. Dynamics in absolute price coordinates. The model above describes dynamics of the order book in *relative* price coordinates, i.e., as a function of the (scaled) distance from the mid-price. The density of the limit order book parameterized by the (absolute) price level $p \in \mathbb{R}$ is given by

$$v_t(p) = u_t(p - S_t), \qquad x \in \mathbb{R}, \tag{1.10}$$

where we extend u_t to $\mathbb{R}$ by setting $u_t(y) = 0$ for $y \in \mathbb{R} \setminus [-L, L]$. Assume S_t follows an (arbitrary) Itô process

$$\mathrm{d}S_t = \theta\mu_t \,\mathrm{d}t + \theta\xi_t^b \,\mathrm{d}W_t^b - \theta\xi_t^a \,\mathrm{d}W_t^a,$$

where $\theta > 0$ and μ_t is predictable and integrable and ξ_t^a and ξ_t^b are predictable and square-integrable processes. This includes the case of price dynamics (1.9), which can be used to express μ_t, ξ_t^a, ξ_t^b in terms of u_t and model parameters. We will not go into such detail here but will return to this in the examples in sections 3 and 4. Define

$$\hat{\xi}_t := \sqrt{(\xi_t^b)^2 + (\xi_t^a)^2 - 2\varrho_{a,b}\xi_t^b\xi_t^a}, \qquad t \geq 0.$$

Using a (generalized) Itô–Wentzell formula (see Appendix A), we can show that v is the solution of a *stochastic moving boundary problem* (Mueller, 2018):

$$\begin{aligned}\mathrm{d}v_t(p) = \Big[&(\eta_a + \tfrac{1}{2}\theta^2\hat{\xi}_t^2)\Delta v_t(p) \\ &+ \Big(\beta_a - \theta\mu_t - \theta\sigma_a(\varrho_{a,b}\xi_t^b - \xi_t^a)\Big)\nabla v_t(p) + \alpha_a v_t(p)\Big]\,\mathrm{d}t \\ &+ (\sigma_a v_t(p) + \theta\xi_t^a\nabla v_t(p))\,\mathrm{d}W_t^a - \theta\xi_t^b\nabla v_t(p)\,\mathrm{d}W_t^b,\end{aligned} \tag{1.11}$$

for $p \in (S_t, S_t + L)$, and

$$\begin{aligned}\mathrm{d}v_t(p) = \Big[&(\eta_b + \tfrac{1}{2}\theta^2\hat{\xi}_t^2)\Delta v_t(p) \\ &+(-\theta\mu_t - \beta_b - \theta\sigma_b(\xi_t^b - \varrho_{a,b}\xi_t^a))\nabla v_t(x) + \alpha_b v_t(p)\Big]\,\mathrm{d}t \\ &+ \theta\xi_t^a\nabla v_t(p)\,\mathrm{d}W_t^a + \Big(\sigma_b v_t(p) - \theta\xi_t^b\nabla v_t(p)\Big)\,\mathrm{d}W_t^b\end{aligned} \tag{1.12}$$

for $x \in (S_t - L, S_t)$ with the moving boundary conditions

$$v_t(S_t) = 0, \qquad v_t(y) = 0 \qquad \forall y \in \mathbb{R} \setminus (S_t - L, S_t + L). \tag{1.13}$$

We refer to (1.13) as a *stochastic boundary condition* at S_t.

Here, we assumed for simplicity that f^a, $f^b = 0$. A more detailed discussion of this result is given in Appendix A.

1.5. Linear evolution models for order book dynamics. We will now describe a more general class of linear models for order book dynamics, rich enough to cover the examples we discussed so far, but also covering all level-1 models where the best bid and ask queue are modeled by positive semimartingales. Generally, the densities of orders in the bid and ask side will take values in some function spaces H^b and H^a, respectively. We assume that orders at relative price level x for $|x| \geq L \in (0, \infty]$ will be canceled. The relative price levels are on the bid side $I^b := (-L, 0)$ and on the ask side $I^a := (0, L)$. Then, in order to preserve the interpretation of a density it will be reasonable to ask $H^b \subset L^1_{loc}(I^b)$ and $H^a \subset L^1_{loc}(I^a)$. From the mathematical side, we will assume that H^a and H^b are real separable Hilbert spaces. For notational convenience we now also set $I := I^b \cup I^a$.

The density of limit orders at relative price level x and time t is given by $u\colon I \times [0, \infty) \times \Omega \to \mathbb{R}$ such that $u^\star := u|_{I^\star}$ is an $H^\star$-valued adapted process. The initial state is described by $h\colon I \to \mathbb{R}$ such that $h^\star := h|_{I^\star}$ is an element in $H^\star$. The (averaged) intrabook dynamics are modeled by linear operators $A_\star\colon \mathrm{dom}(A_\star) \subset H^\star \to H^\star$, for $\star \in \{a, b\}$, which we assume to be densely defined and such that for $\star \in \{a, b\}$ there exist weak solutions in $H^\star$ of the equations

$$\tfrac{\partial}{\partial t} g^\star_t(h^\star) = A_\star g^\star_t(h^\star), \quad t > 0, \quad g^\star_0(h^\star) = h^\star, \tag{1.14}$$

for each initial state $h^\star \in H^\star$.

The random order arrivals and cancellations are assumed to be proportional and are modeled by càdlàg semimartingales X^b and X^a, which we assume to have jumps greater than -1 almost surely. We assume the initial order book state is denoted by $h \in H$ and we write $h^a := h|_{I^a}$, $h^b := h|_{I^b}$.

Model 1.4 (linear homogeneous evolution). *The general form of the linear homogeneous model is*

$$\begin{cases} \mathrm{d}u^b_t = A_b u^b_{t-}\ \mathrm{d}t + u^b_{t-}\ \mathrm{d}X^b_t & \text{on } I^b, \\ \mathrm{d}u^a_t = A_a u^a_{t-}\ \mathrm{d}t + u^a_{t-}\ \mathrm{d}X^a_t & \text{on } I^a, \end{cases} \tag{1.15}$$

for $t \geq 0$, and $u_0 = h$. u can be alternatively expressed as

$$u_t = g^b_t(h^\star)\mathcal{E}_t(X^b)\mathbf{1}_{I^b} + g^a_t(h^\star)\mathcal{E}_t(X^a)\mathbf{1}_{I^a}, \tag{1.16}$$

where g^b and g^a are solutions of (1.14); see Theorem 2.5 below. If, in addition, $t \mapsto \nabla g^b_t(0-)$ and $t \mapsto \nabla g^a_t(0+)$ are of bounded variation, then we obtain the price dynamics (1.9).

Corollary 1.5. *Assume the setting of Model 1.4 and, in addition, that $h^\star$ is an eigenfunction of $-A_\star$ with eigenvalue $\nu_\star \in \mathbb{R}$ for $\star = b$ and $\star = a$. Then, (1.14) can be solved explicitly and*

$$u_t = h^b e^{-\nu_b t}\mathcal{E}_t(X^b)\mathbf{1}_{I^b} + h^a e^{-\nu_a t}\mathcal{E}_t(X^a)\mathbf{1}_{I^a}. \tag{1.17}$$

Remark 1.6. In case that X^b and X^a are (local) martingales, the eigenvalues $-\nu_b$ and $-\nu_a$ play the role of net order arrival rates on the bid and the ask side, respectively.

Model 1.7 (linear models with source terms). *A more realistic setting assumes in addition an influx/outflow of orders at a rate $f^a(x), f^b(x)$ which depends on the distance x to the mid-price* (Cont, Stoikov, and Talreja, 2010). *The equation then becomes*

$$\begin{cases} \mathrm{d}u_t^b = \left(A_b u_t^b + f^b\right)\,\mathrm{d}t + u_t^b\,\mathrm{d}X_t^b & \text{on } I^b, \\ \mathrm{d}u_t^a = \left(A_a u_t^a + f^a\right)\,\mathrm{d}t + u_t^a\,\mathrm{d}X_t^a & \text{on } I^a, \end{cases} \tag{1.18}$$

for $t \geq 0$, with initial condition $u_0 = h$.

As we will discuss in section 4, an interesting case is when f^b (resp., f^a) is an eigenfunction of $-A^b$ (resp., $-A^a$) associated with some eigenvalue ν_b (resp., ν_a). Then by Theorem 2.10 we obtain

$$u_t = \left(g_t^b(h^b - f^a)\mathcal{E}_t(X^b) + f^b Z_t^b\right)\mathbf{1}_{I^b} + \left(g_t^a(h^a - f^a) + f^a Z_t^a\right)\mathbf{1}_{I^a}, \tag{1.19}$$

where, for $\star \in \{a, b\}$, $Z_t^\star$ is the solution of

$$\mathrm{d}Z_t^\star = (1 - \nu_\star Z_{t-}^\star)\,\mathrm{d}t + Z_{t-}^\star\,\mathrm{d}X_t^\star, \quad t \geq 0, \quad Z_0^\star = 1. \tag{1.20}$$

Remark 1.8. If ν_b, $\nu_a > 0$ the state of the order book is mean reverting to the state $f^b 1_{[-L,0)} + f^a\, 1_{(0,L]}$. We will give an example of such a mean-reverting order book model in section 4.

Remark 1.9. Any model for the dynamics of the order book implies a model for price dynamics via (1.9). In particular this implies a relation between price volatility and parameters describing order flow, in the spirit of Cont and de Larrard (2013). We will derive this relation for the examples studied in what follows and use it to construct a model-based intraday volatility estimator.

In the next section, we will study this class of models from a mathematical point of view. We will then continue with the analysis of the two examples mentioned above in sections 3 and 4.

2. Linear stochastic PDE models with multiplicative noise.

In order to further study the properties of the SPDE model (1.2), we require a more explicit characterization of the solution, in order to compute various quantities of interest and estimate model coefficients from observations. A useful approach is to look for a finite-dimensional realization of the infinite-dimensional process u.

Definition 2.1 (finite-dimensional realizations). *A process $u = (u_t)_{t\geq 0}$ taking values in an (infinite-dimensional) function space E is said to admit a finite-dimensional realization of dimension $d \in \mathbb{N}$ if there exists an $\mathbb{R}^d$-valued stochastic process $Z = (Z^1, \ldots, Z^d)$ and a map*

$$\phi : \mathbb{R}^d \to E \quad \text{such that} \quad \forall t \geq 0, \quad u_t = \phi(Z_t).$$

Availability of a finite-dimensional realization for the SPDE (1.2) makes simulation, computation, and estimation problems more tractable, especially if the process Z is a low-dimensional Markov process. Existence of such finite-dimensional realizations for SPDEs have been investigated for SPDEs arising in filtering (Lévine, 1991) and interest rate modeling (Filipovic and Teichmann, 2003; Gaspar, 2006).

We will now show that finite-dimensional realizations may indeed be constructed for a class of SPDEs which includes (1.2), and we use this representation to perform an analytical study of these models.

2.1. Homogeneous equations. We now consider a more general class of linear homogeneous evolution equations with multiplicative noise taking values in a real separable Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. Typically, H will be a function space such as $L^2(I)$ for some interval $I \subset \mathbb{R}$. We consider the following class of evolution equations:

$$\begin{aligned} \mathrm{d}u_t &= Au_{t-}\,\mathrm{d}t + u_{t-}\,\mathrm{d}X_t, \qquad t > 0, \\ u_0 &= h_0 \in H, \end{aligned} \tag{2.1}$$

where X is a real càdlàg semimartingale whose jumps satisfy $\Delta X_t > -1$ a.s. and $A\colon \mathrm{dom}(A) \subset H \to H$ a linear operator on H whose adjoint we denote by A^*. We assume that $\mathrm{dom}(A) \subset H$ is dense and A is closed. Since A is closed we have that also $\mathrm{dom}(A^*) \subset H$ is dense and that $A^{**} = A$ (Yosida, 1995, Theorem VII.2.3).

Definition 2.2. *An adapted H-valued stochastic process (u_t) is an* (analytical) weak solution *of* (2.1) *with initial condition h_0 if, for all $\varphi \in \mathrm{dom}(A^*)$, $[0,\infty) \ni t \mapsto \langle u_t, \varphi \rangle_H \in \mathbb{R}$ is càdlàg a.s. and for each $t \geq 0$, a.s.*

$$\langle u_t, \varphi \rangle_H - \langle h_0, \varphi \rangle_H = \int_0^t \langle u_{s-}, A^*\varphi \rangle_H \,\mathrm{d}s + \int_0^t \langle u_{s-}, \varphi \rangle_H \,\mathrm{d}X_s.$$

The case $X \equiv 0$ corresponds to a notion of weak solution for the PDE:

$$\forall t > 0, \tfrac{\partial}{\partial t} g_t = A g_t, \qquad g_0 = h_0. \tag{2.2}$$

That is, for all $\varphi \in \mathrm{dom}(A^*)$,

$$\langle g_t, \varphi \rangle_H - \langle h_0, \varphi \rangle_H = \int_0^t \langle g_s, A^*\varphi \rangle_H \,\mathrm{d}s, \tag{2.3}$$

where the integral on the right-hand side is assumed to exist.[2] In particular, this yields that $[0,\infty) \ni t \mapsto \langle g_t, \varphi \rangle_H \in \mathbb{R}$ is continuous.

Remark 2.3. By considering bid and ask side separately, we can bring (1.2) into the form of (2.1), where X is a Brownian motion and A is given by $A := \eta\Delta \pm \beta\nabla + \alpha\,\mathrm{Id}$ on $H := L^2(I)$, $I := (0, L)$, or $I := (-L, 0)$, with domain

$$\mathrm{dom}(A) := H^2(I) \cap H_0^1(I),$$

where $H_0^1(I)$ is the closure in $H^1(I)$ of test functions with compact support in I.

Denote by $Z_t = \mathcal{E}_t(X)$ the stochastic exponential of X. We recall the following useful lemma (see, e.g., Karatzas and Kardaras (2007, Lemma 3.4)).

[2]Note that this slightly differs from the classical formulation of weak solutions for PDEs.

Lemma 2.4. *Let*

$$Y_t := -X_t + [X,X]^c_t + \sum_{s\le t} \frac{(\Delta X_s)^2}{1+\Delta X_s}, \qquad t \ge 0, \tag{2.4}$$

Then, $\mathcal{E}_t(X)\mathcal{E}_t(Y) = 1$ *almost surely for all* $t \ge 0$. *Moreover,*

$$[X,Y] = -[X,X]^c - \sum_{s\le \cdot} \frac{(\Delta X_s)^2}{1+\Delta X_s}. \tag{2.5}$$

Theorem 2.5. *Let* $Z := \mathcal{E}(X)$, $h_0 \in H$. *Then every weak solution of* (2.1) *is of the form*

$$u_t := Z_t g_t, \qquad t \ge 0,$$

where g *is a weak solution of* (2.2).

Remark 2.6. In particular, the SPDE (2.1) admits a two-dimensional realization in the sense of Definition 2.1 with factor process $(t, \mathcal{E}_t(X))$ and $\phi(t,y) := y g_t$.

Proof. Set $u_t := g_t Z_t$, $t \ge 0$, and for $\varphi \in D(A^*)$ write $B^\varphi_t := \langle g_t, \varphi\rangle_H$, $C^\varphi_t := B^\varphi_t Z_t = \langle u_t, \varphi\rangle_H$. Since $t \mapsto \langle g_t, \varphi\rangle_H$ is continuous and Z is scalar and càdlàg, we get that $t \mapsto \langle u_t, \varphi\rangle_H$ is càdlàg. Note that B^φ is of finite variation and Z is a semimartingale, so that also C^φ is a semimartingale. Moreover, by the Itô product rule and since B^φ is of finite variation and continuous,

$$\mathrm{d}C^\varphi_t = B^\varphi_t \,\mathrm{d}Z_t + Z_{t-}\,\mathrm{d}B^\varphi_t = B^\varphi_{t-} Z_{t-}\,\mathrm{d}X_t + \langle u_{t-}, A^*\varphi\rangle_H \,\mathrm{d}t, \tag{2.6}$$

which is (2.1). Now, let u be a solution of (2.1) and set

$$Y := -X + [X,X]^c + J, \qquad J := \sum_{s\le\cdot} \frac{(\Delta X_s)^2}{1+\Delta X_s},$$

and $Z_t := \mathcal{E}_t(Y)$, $t \ge 0$. Recall that by Lemma 2.4 we have $Z_t \mathcal{E}_t(X) = 1$ for all $t \ge 0$. Set $g_t := Z_t u_t$, and, as above, fix $\varphi \in \mathrm{dom}(A^*)$ and write $B^\varphi_t := \langle u_t Z_t, \varphi\rangle_H = \langle g_t, \varphi\rangle_H$ and $C^\varphi_t := \langle u_t, \varphi\rangle_H$. By Itô's product rule and Lemma 2.4,

$$\begin{aligned}
\mathrm{d}B^\varphi_t &= C^\varphi_{t-}\,\mathrm{d}Z_t + Z_21t-\,\mathrm{d}C^\varphi_t + \mathrm{d}[C^\varphi, Z]_t \\
&= C^\varphi_{t-} Z_{t-}\,\mathrm{d}Y_t + Z_{t-}\langle u_{t-}, A^*\varphi\rangle_H \,\mathrm{d}t + C^\varphi_{t-} Z_{t-}\,\mathrm{d}X_t + C^\varphi_{t-} Z_{t-}\,\mathrm{d}[X,Y]_t \\
&= \langle g_{t-}, A^*\varphi\rangle_H \,\mathrm{d}t + B^\varphi_{t-}\left(\mathrm{d}[X,X]^c_t + \mathrm{d}J_t\right) - B^\varphi_{t-}\left(\mathrm{d}[X,X]^c_t + \mathrm{d}J_t\right) \\
&= \langle g_{t-}, A^*\varphi\rangle_H \,\mathrm{d}t.
\end{aligned}$$

Thus, g is a weak solution of (2.2). ■

Example 2.7. Let A be the generator of a strongly continuous semigroup $(S_t)_{t\ge 0}$. Then, for $h_0 \in H$ define

$$g_t := S_t h_0, \quad t \ge 0,$$

which is a weak solution of (2.2). By Theorem 2.5.

$$u_t := \mathcal{E}_t(X) S_t h_0, \qquad t \ge 0,$$

is a weak solution of (2.1).

Remark 2.8. If h_0 is an eigenfunction of A with eigenvalue ν, then $g_t = e^{\nu t} h_0$ is the unique locally H-integrable solution of (2.2), and the unique solution of (2.1) is given by

$$u_t := h_0 e^{\nu t} \mathcal{E}_t(X).$$

2.2. Inhomogeneous equations. We keep the assumptions on A, h_0, and X from the previous section and let $f \in H$. We now consider the inhomogeneous linear evolution equations

$$\begin{aligned} \mathrm{d}u_t &= [Au_t + \alpha f]\,\mathrm{d}t + u_{t-}\,\mathrm{d}X_t, \qquad t \geq 0, \\ u_0 &= h_0. \end{aligned} \tag{2.7}$$

Definition 2.9. *A weak solution of* (2.7) *is an adapted H-valued stochastic process u such that for all $\varphi \in \mathrm{dom}(A^*)$ the mapping $[0,\infty) \ni t \mapsto \langle u_t, \varphi\rangle_H$ is càdlàg and*

$$\langle u_t, \varphi\rangle_H - \langle h_0, \varphi\rangle_H = \int_0^t \langle u_{s-}, A^*\varphi\rangle_H \,\mathrm{d}s + \int_0^t \langle u_{s-}, \varphi\rangle_H \,\mathrm{d}X_s + t\alpha\langle f, \varphi\rangle_H, \quad t \geq 0,$$

almost surely.

We exclude the cases $\alpha = 0$ or $f \equiv 0$ which correspond to the homogeneous case discussed above. Let us first consider the case where A admits at least one eigenfunction.

Theorem 2.10. *Suppose that $f \in \mathrm{dom}(A)$ is an eigenfunction for A with eigenvalue $\lambda \in \mathbb{R}$, and let $z_0 > 0$ and Z be the solution of*

$$\mathrm{d}Z_t = (\lambda Z_{t-} + \alpha)\,\mathrm{d}t + Z_{t-}\,\mathrm{d}X_t, \quad t \geq 0, \quad Z_0 = z_0. \tag{2.8}$$

Then,

(i) *the stochastic process defined by $u_t = Z_t f$, $t \geq 0$, is a solution of* (2.7) *with initial condition $h_0 := z_0 f$;*

(ii) *let, in addition, $h_0 \in H$ be such that there exists a weak solution $g = (g_t)_{t\geq 0}$ of the deterministic equation*

$$\tfrac{\partial}{\partial t} g_t = A g_t,\ t \geq 0, \qquad g_0 = h_0 - z_0 f, \tag{2.9}$$

and then, $u_t := g_t \mathcal{E}_t(X) + f Z_t$ is a solution of (2.7) *with initial condition h_0;*

(iii) *let $h_0 \in H$ be such that there exists a weak solution $u = (u_t)_{t\geq 0}$ of* (2.7) *with initial condition h_0. Then, $g := (u - fZ)\mathcal{E}(X)^{-1}$ is a weak solution of* (2.9).

Remark 2.11. Let $(Z^1_t)_{t\geq 0}$ and $(Z^2_t)_{t\geq 0}$ be given by (2.8) with respective initial data z_1, $z_2 > 0$, $z_1 \neq z_2$. Then, in fact $Z^2_t - Z^1_t = (z_2 - z_1)\mathcal{E}_t(X)$, which is consistent with choosing different values for z_0 in (ii).

Proof. Part (i) follows by direct a computation: Let $\varphi \in H$; then for $t \geq 0$,

$$\begin{aligned} d\langle u_t, \varphi\rangle_H &= \langle f, \varphi\rangle_H\, dZ_t \\ &= \langle f, \varphi\rangle_H (\lambda Z_{t-} + \alpha)\, dt + \langle f, \varphi\rangle_H Z_{t-} dX_t \\ &= [\langle u_{t-}, A^*\varphi\rangle_H + \alpha\,\langle f, \phi\rangle_H]\, dt + \langle u_{t-}, \phi\rangle_H\, dX_t. \end{aligned} \tag{2.10}$$

Similarly, we obtain that any solution u of (2.7) with initial data $h_0 \in H$ can be written as

$$u = u^{\circ,(h_0 - z_0 f)} + u^{(z_0 f)},$$

where $u^{\circ,(h_0-z_0 f)}$ is the solution of the homogeneous problem (2.1) with initial data $h_0 - z_0 f$ and $u^{(z_0 f)}$ is a solution of (2.7) with initial data $z_0 f$. Then, part (i) and Theorem 2.5 finish the proof of (ii) and (iii). ∎

It is then readily verified using Itô's formula that the unique solution of (2.8) is given by

$$Z_t := \mathcal{E}_t(X) e^{\lambda t} \left(Z_0 + \alpha \int_0^t e^{-\lambda s} \mathcal{E}_{s-}(Y)\, \mathrm{d}s \right), \qquad t \geq 0, \tag{2.11}$$

where

$$Y_t := -X_t + [X, X]_t^c + \sum_{s \leq t} \frac{\Delta X_s^2}{1 + \Delta X_s}, \qquad t \geq 0.$$

We now focus on the case $X = \sigma W$ for a real Brownian motion W and a constant $\sigma > 0$. Then, we will consider *regular* two-dimensional realizations of the form $u_t = \Phi(t, Y_t)$, where

(a) Y is a diffusion process with state space $J \subseteq \mathbb{R}$, satisfying

$$\mathrm{d}Y_t = b(Y_t)\, \mathrm{d}t + a(Y_t)\, \mathrm{d}W_t,$$

for measurable functions $b, a \colon J \to \mathbb{R}$, where J has nonempty interior, $a(y) > 0$ for all $y \in J$, and $1/a$ is locally integrable on J;

(b) $\Phi \colon [0, \infty) \times J \to \mathrm{dom}(A)$ such that for all $\varphi \in \mathrm{dom}(A^*)$, the maps defined by $\Phi^\varphi(t, y) := \langle \Phi(t, y), \varphi \rangle$, $t \geq 0$, $y \in J$, are in $C^{1,2}(\mathbb{R}_{\geq 0} \times J; \mathbb{R})$.

Examples of such regular two-dimensional realizations are given by Theorem 2.10(i).

Theorem 2.12. *Let $X_t = \sigma W_t$, $t \geq 0$, for $\sigma > 0$ and a real Brownian motion W, and assume that (2.7) admits a regular finite-dimensional realization $u_t = \Phi(t, Y_t)$, $t \geq 0$. Then f is an eigenfunction of A for some eigenvalue $\lambda \in \mathbb{R}$, and there exists an invertible transformation $h \colon J \to \mathbb{R}_+$ such that for $t \geq 0$, almost surely*

$$Z_t = h(Y_t), \qquad u_t = \Phi(t, h^{-1}(Z_t)) = f Z_t,$$

where Z is given by (2.8).

Proof. Let $\varphi \in \mathrm{dom}(A^*)$, and

$$\Phi^\varphi(t, Y_t) := \langle \Phi(t, Y_t), \varphi \rangle\,. \tag{2.12}$$

An application of the Itô formula yields

$$\begin{aligned} \mathrm{d}\langle u_t, \varphi \rangle &= d\Phi^\varphi(t, Y_t) \\ &= \left(\partial_t \Phi^\varphi(t, Y_t) + \partial_y b(Y_t) \Phi^\varphi(t, Y_t) + \frac{1}{2} a^2(Y_t) \partial_{yy} \Phi^\varphi(t, Y_t) \right) dt \\ &\quad + a(Y_t) \partial_y \Phi^\varphi(t, Y_t) dW_t. \end{aligned} \tag{2.13}$$

Comparing the martingale term with (2.7), we see that Φ^φ satisfies the ODE

$$\partial_y \Phi^\varphi(t, Y_t) = \frac{\sigma \Phi^\varphi(t, Y_t)}{a(Y_t)},$$

$dt \otimes d\mathbb{P}$-a. e., and hence Φ^φ must be of the form

$$\Phi^\varphi(t, y) = g^\varphi(t) h(y) = g^\varphi(t) \exp\left(\int_{y_0}^{y} \frac{\sigma d\eta}{a(\eta)}\right), \qquad t \geq 0,\ y \in J,$$

for some $g^\varphi \in C^1(\mathbb{R}_{\geq 0})$ and y_0 in the interior of J. The regularity property of the representation guarantees that h is well-defined and strictly monotone increasing. We stress that h is in fact independent of $\varphi \in \operatorname{dom}(A^*)$. Setting $Z_t = h(Y_t)$, we see that Z satisfies

$$dZ_t = m(Z_t)dt + \sigma Z_t dW_t$$

for the drift function $m = (bh') \circ h^{-1} + \frac{1}{2}(a^2 h'') \circ h^{-1}$.

Note that for each $t \geq 0$, the mapping $\varphi \mapsto g^\varphi(t)$ is linear continuous from $\operatorname{dom}(A^*) \subset H$ into $\mathbb{R}$. Since $\operatorname{dom}(A^*) \subset H$ is dense, by the Riesz representation theorem for each $t \geq 0$ there exists $g(t) \in H$ such that

$$\langle g(t), \varphi \rangle = g^\varphi(t). \tag{2.14}$$

Since $\Phi^\varphi(t, y) = g^\varphi(t) h(y)$, g^φ is differentiable and (2.13) becomes, for $\varphi \in \operatorname{dom}(A^*)$,

$$d \langle u_t, \varphi \rangle = (Z_t \partial_t g^\varphi(t) + g^\varphi(t) m(Z_t))\, dt + g^\varphi(t) Z_t dW_t.$$

Comparing the drift terms with (2.7) yields for $t \geq 0$, $\varphi \in \operatorname{dom}(A^*)$, and $z \in h(J)$,

$$z \left(\langle g(t), A^* \varphi \rangle - \partial_t g^\varphi(t)\right) + \alpha \langle f, \varphi \rangle = m(z) g^\varphi(t). \tag{2.15}$$

Evaluating at two different points $z_0, z_1 \in h(J)$ and subtracting we obtain that

$$\left(\langle g(t), A^* \varphi \rangle - \partial_t \langle g(t), \varphi \rangle\right) \cdot (z_1 - z_0) = \langle g(t), \varphi \rangle \cdot (m(z_1) - m(z_0))$$

for all $t \in \mathbb{R}_{\geq 0}$, $\varphi \in \operatorname{dom}(A^*)$, and $z_0, z_1 \in h(J)$. We conclude that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\langle g(t), A^*\varphi\rangle - \partial_t \langle g(t), \varphi\rangle = \lambda \langle g(t), \varphi\rangle \tag{2.16}$$

and

$$m(z_1) - m(z_0) = \lambda(z_1 - z_0). \tag{2.17}$$

Thus m must be of the form $m(z) = \lambda z + c$ for $c := m(0)$. Inserting into (2.15) we obtain that

$$\alpha \langle f, \varphi\rangle = c \langle g(t), \varphi\rangle \qquad \forall \varphi \in \mathrm{dom}(A^*).$$

Since $\mathrm{dom}(A^*) \subset H$ is dense the equation holds for all $\varphi \in H$. Due to the assumption that $\alpha \neq 0$ and f is nonzero, also $c \neq 0$ and we get $g(t) = \frac{\alpha}{c} f$. In particular, $g(t)$ is independent of t and (2.16) yields

$$\langle f, A^*\varphi\rangle = \left\langle \tfrac{\lambda c}{\alpha} f, \varphi\right\rangle \qquad \forall \varphi \in \mathrm{dom}(A^*).$$

This means that $f \in \mathrm{dom}(A^{**})$. Since $A = A^{**}$ (see, e.g., Yosida (1995, Theorem VII.2.3)), we have $f \in \mathrm{dom}(A)$ and

$$\langle f, A^*\varphi\rangle = \langle Af, \varphi\rangle = \lambda\langle f, \varphi\rangle \qquad \forall \varphi \in \mathrm{dom}(A^*).$$

By density of $\mathrm{dom}(A^*)$ in H this yields that $Af = \lambda f$, i.e., f must be an eigenfunction of A with eigenvalue λ.

Putting everything together, we have shown that $u_t = \frac{\alpha}{c} f Z_t$ where

$$dZ_t = (\lambda Z_t + c)\, dt + \sigma Z_t dW_t.$$

Rescaling Z by $\frac{\alpha}{c}$ concludes the proof. ■

2.3. Linear SDEs and Pearson diffusions. Let again $X_t = \sigma W_t$ for some $\sigma > 0$ and a real Brownian motion W. The factor processes Z appearing above are then special cases of the linear SDE

$$dZ_t = (aZ_t + c)dt + (bZ_t + d)dW_t, \quad t \geq 0, \quad Z_0 = z_0, \tag{2.18}$$

studied, e.g., in Kloeden and Platen (1992, Chapter 4) or Kallenberg (2002, Proposition 21.2). Well-known special cases are the geometric Brownian motion ($c = d = 0$) and the Ornstein–Uhlenbeck process ($b = 0$). Relevant in our context is the less common case $d = 0$, on which we focus now. Using (2.11), the solution is given by

$$Z_t = X_t \left(Z_0 + c \int_0^t X_s^{-1} ds \right), \quad t \geq 0, \tag{2.19}$$

where

$$X_t = \exp\left(\left(a - \frac{b^2}{2} \right) t + bW_t \right), \quad t \geq 0. \tag{2.20}$$

Solutions of (2.18) have also been studied in the context of reciprocal gamma diffusions (see, e.g., Case 4 in Forman and Sørensen (2008)) or also Pearson diffusions. These are generalizations of (2.18) that allow for a square-root term in the diffusion coefficient.

Proposition 2.13. *Assume that $z_0 > 0$, $a < 0$, and $c > 0$. Then, Z has unique invariant distribution ϖ, which is an inverse Gamma distribution with shape parameter $1-\frac{2a}{b^2}$ and scale parameter $\frac{b^2}{2c}$ and, for any bounded measurable function $\phi\colon (0,\infty) \to \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{E}\left[\phi(Z_t)\right] = \lim_{t\to\infty} \frac{1}{t}\int_0^t \phi(Z_s)\, \mathrm{d}s = \int_0^\infty \phi(x)\varpi(\,\mathrm{d}x).$$

Proof. First, note that

$$s'(x) := x^{-2\frac{a}{b^2}} e^{2\frac{c}{b^2 x}}, \qquad m(\,\mathrm{d}x) := x^{2(\frac{a}{b^2}-1)} e^{-2\frac{c}{b^2 x}}\, \mathrm{d}x, \quad x \in (0,\infty),$$

define a scale density and speed measure for Z. Then, one can easily verify that Z is strictly positive and recurrent on $(0,\infty)$; see, e.g., Karatzas and Shreve (1987, Proposition 5.5.22). Moreover, $m((0,\infty)) < \infty$ and so the unique invariant distribution of Z is

$$\varpi(A) := \frac{m(A)}{m((0,\infty))}. \tag{2.21}$$

The remaining results then follow from, e.g., Borodin and Salminen (2012, II.35) or Revuz and Yor (1999, X.3.12). ∎

$\mu(t) := \mathbb{E}Z_t$, $t \geq 0$, satisfies the ODE

$$\tfrac{\partial}{\partial t}\mu(t) = a\mu(t) + c,\ t > 0, \qquad \mu(0) = Z_0.$$

Thus,

$$\mu(t) = \left(Z_0 + \frac{c}{a}\right) e^{at} - \frac{c}{a}. \tag{2.22}$$

Remark 2.14. Let $a < 0$, $c > 0$, and (Z_t) be the stationary solution of

$$\mathrm{d}Z_t = (aZ_t + c)\, \mathrm{d}t + bZ_t\, \mathrm{d}W_t,$$

that is, Z_0 is chosen distributed according to inverse gamma distribution with shape parameter $1-\frac{2a}{b^2}$ and scale parameter $\frac{b^2}{2c}$. Then, as shown in Bibby, Skovgaard, and Sørensen (2005), the autocorrelation function of (Z_t) is given by

$$r(t) := \mathrm{Corr}(Z_{s+t}, Z_s) = e^{at}, \qquad s, t \geq 0. \tag{2.23}$$

To study price dynamics it is also useful to examine the reciprocal process $Y = 1/Z$. When $d = 0$, $Y = 1/Z$ is the unique solution of

$$\mathrm{d}Y_t = -Y_t(a - b^2 + cY_t)\, \mathrm{d}t - bY_t\, \mathrm{d}W_t, \qquad Y_0 = z_0^{-1}. \tag{2.24}$$

In particular, with X given in (2.20),

$$Y_t = \mathcal{E}_t(-bW_{\cdot} - a(.)) \left(Z_0 + c\int_0^t X_s^{-1}\, \mathrm{d}s\right)^{-1}, \quad t \geq 0. \tag{2.25}$$

When $a < b^2$, (2.24) is called the stochastic logistic equation.

2.4. Positivity, stationarity, and martingale property. Let us first come back to the linear homogeneous situation. On average, market makers do not accumulate inventory, which suggests considering the baseline case of *balanced order flow* for which X is a (local) martingale. If X is a local martingale with $\Delta X > -1$ a.s., then, from the properties of stochastic exponentials, we obtain that

- the weak solution u_t of the homogeneous equation (2.1) is a local martingale if and only if the initial condition h_0 is A-harmonic: $h_0 \in \mathrm{dom}(A)$ and $Ah_0 = 0$;
- if $\mathcal{E}(M)$ is a martingale and $Ah_0 = 0$, then $(u_t)_{t\geq 0}$ is a martingale.

In the Brownian motion case, from the discussion in the previous section we directly obtain the following.

Corollary 2.15. *Let $X = \sigma W$ where W is a standard Brownian motion and $\sigma > 0$, and let u be the solution of the inhomogeneous equation (2.7), where f is an eigenfunction of A with eigenvalue $-\nu$ and $h_0 = z_0 f$, for some $z_0 > 0$. If $\nu > 0$ and $\alpha > 0$, then*

$$u_t \overset{t\to\infty}{\Rightarrow} \quad f Z_\infty,$$

where Z_∞ has an inverse Gamma distribution with shape parameter $1 + 2\frac{\nu}{\sigma^2}$ and scale parameter $\frac{\sigma^2}{2\alpha}$.

Remark 2.16. The inverse Gamma distribution has a Pareto (right) tail with tail index $1 + 2\nu$ in this case: the kth moment of $\mathbb{E}(Z_\infty^k) < \infty$ if and only if $k < 1 + 2\nu$.

So far, we have set aside the positivity constraint for u. By Theorem 2.5 this reduces to analysis of the deterministic equation. In the case of second-order elliptic operators, positivity results from the comparison principle, whenever the initial condition h_0 is positive.

Assumption 2.17. Let $I \subset \mathbb{R}$ be an interval and suppose that A is a uniformly elliptic operator of the form

$$Au(x) = \eta(x)\Delta u(x) + \beta(x)\nabla u(x) + \alpha(x)u(x), \qquad x \in I,$$

with Dirichlet boundary conditions, and where η, β, and α are smooth and bounded coefficients, and in particular $\eta(x) \geq \underline{\eta} > 0$ for all $x \in I$.

In addition, the principal eigenvalue of A, λ_1 has an eigenfunction f which is positive on I (Evans, 2010, section 6.5). Note that the factor process Z_t has state space $(0, \infty)$. We thus obtain the following corollary.

Corollary 2.18 (positivity). *Under Assumption 2.17,*

(i) *if h_0 is positive on I, then the solution g_t of (2.2) and the solution u_t of (2.1) are a.s. positive on I;*

(ii) *if f is the principal eigenfunction of A, then the finite-dimensional realization $u_t = fZ_t$ of (2.7) is a.s. positive on I.*

This simple result thus guarantees the existence of a solution with the correct sign, thereby avoiding recourse to "reflected" solutions as in Hambly, Kalsi, and Newbury (2020) and considerably simplifying the analysis of our model.

3. A two-factor model.

We now study the simplest example of model satisfying Assumption 2.17, namely the case of constant coefficients η_a, η_b, σ_a, $\sigma_b > 0$, β_a, $\beta_b \geq 0$, α_a, $\alpha_b \in \mathbb{R}$,

$$\begin{aligned} \mathrm{d}u_t(x) &= [\eta_a \Delta u_t(x) + \beta_a \nabla u_t(x) + \alpha_a u_t(x)]\,\mathrm{d}t + \sigma_a u_t(x)\,\mathrm{d}W_t^a, \quad x \in (0, L), \\ \mathrm{d}u_t(x) &= [\eta_b \Delta u_t(x) - \beta_b \nabla u_t(x) + \alpha_b u_t(x)]\,\mathrm{d}t + \sigma_b u_t(x)\,\mathrm{d}W_t^b, \quad x \in (-L, 0), \\ u_t(x) &= 0, \quad x \in \{-L, 0, L\}, \qquad u_0 \in L^2(-L, L), \end{aligned} \tag{3.1}$$

together with the sign condition

$$u_t(x) \leq 0, \quad x \in (-L, 0), \quad \text{and} \quad u_t(x) \geq 0, \quad x \in (0, L),\ t \geq 0. \tag{3.2}$$

In the following, we will write $u_0^b := u_0|_{[-L,0]}$ and $u_0^a := u_0|_{[0,L]}$.

3.1. Spectral representation of solutions.

A spectral representation of the operator may be used to obtain an analytical solution to this model.

Proposition 3.1. *Let $I = (-L, 0)$ or $I = (0, L)$ and $\eta > 0$, β, $\alpha \in \mathbb{R}$, and consider the linear operator*

$$A := \eta\Delta + \beta\nabla + \alpha\,\mathrm{Id} \tag{3.3}$$

on $L^2(I)$ with $\mathrm{dom}(A) := \left\{u \in H^2(I) \middle| u|_{\partial I} = 0\right\} = H^2(I) \cap H_0^1(I)$. The eigenvalues of $-A$ are real and given by

$$\nu_k = -\alpha + \frac{\eta k^2 \pi^2}{L^2} + \frac{\beta^2}{4\eta}, \quad k = 1, 2, \ldots, \tag{3.4}$$

with corresponding eigenfunctions

$$h_k(x) := e^{-\frac{\beta}{2\eta}x} \sin\left(\tfrac{k\pi}{L}x\right), \qquad x \in I.$$

In particular the only positive eigenfunction is h_1.

Proof. First we note that ϕ is an eigenfunction of A with eigenvalue ν if and only if

$$x \mapsto e^{\frac{\beta}{2\eta}x}\phi(x)$$

is an eigenfunction of $A_0 := \eta\Delta + \alpha\,\mathrm{Id}$ with zero Dirichlet boundary conditions for eigenvalue $\nu + \frac{\beta^2}{4\eta}$. Details of calculations are given in Cont (2005). The operator A_0 with domain $\mathrm{dom}(A_0) := \mathrm{dom}(A)$ is self-adjoint and has compact resolvent (Cont, 2005) and eigenvalues

$$\alpha - \frac{\eta k^2 \pi^2}{L^2}, \qquad k \in \mathbb{N}. \tag{3.5}$$

Eigenfunctions of A_0 with eigenvalue $\nu \in \mathbb{R}$ are solutions of the Sturm–Liouville problem

$$\eta g''(x) + (\alpha - \nu)g(x) = 0, \qquad x \in I, \tag{3.6}$$

with zero boundary conditions, which yields that g must be of the form

$$g(x) = ce^{-\gamma_1 x}\sin(\gamma_2 x), \quad \text{where} \quad \gamma_1 = 0, \qquad \gamma_2 = \frac{\nu - \alpha}{\eta}. \tag{3.7}$$

The zero boundary conditions at 0 and $\pm L$ imply $\gamma_2 = \frac{k}{L}\pi$ for some $k \in \mathbb{N}$ so

$$\nu = \alpha - \frac{\eta k^2 \pi^2}{L^2}. \tag{3.8}$$

Translating this from A_0 to A yields the result. ∎

Define the bilinear forms

$$L^2(-L,0) \times L^2(-L,0) \ni (f,g) \mapsto \langle f,g\rangle_{-\gamma} := \frac{2}{L}\int_{-L}^{0} f(x)g(x)e^{-2\gamma x}\,\mathrm{d}x \tag{3.9}$$

and

$$L^2(0,L) \times L^2(0,L) \ni (f,g) \mapsto \langle f,g\rangle_{\gamma} := \frac{2}{L}\int_{0}^{L} f(x)g(x)e^{2\gamma x}\,\mathrm{d}x \tag{3.10}$$

which define equivalent inner products, respectively, for $L^2(-L,0)$ and $L^2(0,L)$. For $\gamma > 0$, and $k \in \mathbb{N}$, define

$$\nu_k^a := -\alpha_a + \frac{\eta_a k^2\pi^2}{L^2} + \frac{\beta_a^2}{4\eta_a}, \quad \nu_k^b := -\alpha_b + \frac{\eta_b k^2\pi^2}{L^2} + \frac{\beta_b^2}{4\eta_b}, \tag{3.11}$$

$$h_k^a(x) := e^{\frac{-\beta_a}{2\eta_a}x}\sin\left(\frac{k\pi}{L}x\right), \qquad x \in (0,L), \tag{3.12}$$

$$h_k^b(x) := e^{\frac{\beta_b}{2\eta_b}x}\sin\left(\frac{k\pi}{L}x\right), \qquad x \in (-L,0). \tag{3.13}$$

Let

$$\gamma_a := \frac{\beta_a}{2\eta_a}, \qquad \gamma_b := \frac{\beta_b}{2\eta_b}. \tag{3.14}$$

Then $(h_k^b)_{k\in\mathbb{N}}$ is an orthonormal basis of $(L^2(-L,0), \langle\cdot,\cdot\rangle_{-\gamma_b})$ and $(h_k^a)_{k\in\mathbb{N}}$ is an orthonormal basis for $(L^2(0,L), \langle\cdot,\cdot\rangle_{\gamma_a})$ and solutions for the SPDE may be constructed using an expansion along these bases.

Proposition 3.2. *Let $u_0 \in L^2(-L,L)$, $u_0^a := u_0|_{[0,L]}$, $u_0^b := u_0|_{[-L,0]}$. Then $(u_t)_{t\geq 0}$ defined by*

$$u_t(x) := \begin{cases} \mathcal{E}_t(\sigma_b W^b)\sum_{k=1}^{\infty} e^{-\nu_k^b t}\left\langle u_0^b, h_k^b\right\rangle_{-\gamma_b} h_k^b(x), & x \in (-L,0), \\ \mathcal{E}_t(\sigma_a W^a)\sum_{k=1}^{\infty} e^{-\nu_k^a t}\left\langle u_0^a, h_k^a\right\rangle_{\gamma_a} h_k^a(x), & x \in (0,L), \\ 0, & x \in \{-L,0,L\}. \end{cases} \tag{3.15}$$

is the unique continuous weak solution of (3.1) in the sense of Definition 2.2.

Proof. The unique continuous solutions of the respective deterministic equations are given by $(S_t^b u_0^b)_{t\geq 0}$ and $(S_t^a u_0^a)_{t\geq 0}$, where $(S_t^b)_{t\geq 0}$ and $(S_t^a)_{t\geq 0}$ are the Dirichlet semigroups generated by

$$A_b = \eta_b \Delta u_t - \beta_b \nabla + \alpha_b \quad \text{and} \quad A_a = \eta_a \Delta + \beta_a \nabla + \alpha_a \tag{3.16}$$

on $(-L, 0)$ and $(0, L)$, respectively. Thus, from Theorem 2.5 we get

$$u_t(x) = \begin{cases} \mathcal{E}_t(\sigma_b W^b) S_t^b u_0^b(x), & x \in (-L, 0), \\ \mathcal{E}_t(\sigma_a W^a) S_t^a u_0^a(x), & x \in (0, L). \end{cases} \tag{3.17}$$

(S_t^a) and (S_t^b) are linear continuous so that for each $h^a \in L^2(0, L)$, $h^b \in L^2(-L, 0)$,

$$S_t^a h^a = \sum_{k\in\mathbb{N}} \langle u_0^a, h_k^a \rangle_{\gamma_a} S_t^a h_k^b, \qquad \text{and} \qquad S_t^b h^b = \sum_{k\in\mathbb{N}} \left\langle u_0^b, h_k^b \right\rangle_{-\gamma_b} S_t^b h_k^b.$$

By Proposition 3.1 h_k^a (resp., h_k^b) are eigenfunctions of A_a (resp., A_b) and thus also of S^a (resp., S^b). This yields the desired representation, where the series converge in L^2. To obtain pointwise convergence, we note that for $x \in [0, L]$ and $t > 0$, by the Cauchy–Schwarz inequality, Parseval's identity, and the integral criterion for sequences, for $\star \in \{a, b\}$,

$$\begin{aligned} \sum_{k=1}^{\infty} \left| e^{-\nu_k^\star t} \left\langle u_0|_{(0,L)}, h_k^\star \right\rangle_{\frac{\beta_\star}{\eta_\star}} h_k^\star(x) \right| &\leq \left\| h|_{(0,L)} \right\|_{L^2(0,L)} \sqrt{\sum_{k=1}^{\infty} e^{-2\nu_k^\star t}} \\ &\leq \left\| u_0|_{(0,L)} \right\|_{L^2(0,L)}^2 e^{t(\alpha_\star - \frac{\beta_\star^2}{4\eta_\star})} \sqrt{\int_0^\infty e^{-2t \frac{\eta_\star \pi^2}{L^2} y^2} \,\mathrm{d}y}. \end{aligned}$$ ■

When $\eta > 0$, then the weights $e^{-\nu_k t}$ of the spectral decomposition decay exponentially in k^2 for large k. This justifies approximating the solution by the first few terms. Note also that the only positive eigenfunctions are the principal eigenfunctions h_1^a and $-h_b^1$ so the sign constraints (3.2) only if the projection of the solution along the principal eigenfunctions dominates the other terms in the expansion. This motivates us to focus on solutions which live in the first eigenspace. This occurs if the initial condition is a (positive) linear combination of h_1^a and h_1^b. We will later show that this assumption is supported by market data. This leads to a finite-dimensional realization which satisfies the sign constraints (3.2).

Corollary 3.3. *Let $V_0^a > 0$, respectively, $V_0^b > 0$, and define*

$$H_1^a(x) = \frac{h_1^a(x)\mathbf{1}_{(0,L)}(x)}{\int_0^L |h_1^a|} \geq 0 \qquad \text{and} \qquad H_1^b(x) = \frac{h_1^b(x)\mathbf{1}_{(-L,0)}(x)}{\int_{-L}^0 |h_1^b|} \leq 0. \tag{3.18}$$

The unique solution of (3.1)–(3.2) with initial condition $u_0 = V_0^a H_1^a + V_0^b H_1^b$ is given by

$$u_t(x) = H_1^b(x) V_t^b + H_1^a(x) V_t^a, \quad t \geq 0, \, x \in [-L, L], \quad \text{where} \tag{3.19}$$

$$\nu^a = -\alpha_a + \frac{\eta_a \pi^2}{L^2} + \frac{(\beta_a)^2}{4\eta_a}, \qquad \nu^b = -\alpha_b + \frac{\eta_b \pi^2}{L^2} + \frac{(\beta_b)^2}{4\eta_b} \quad \text{and} \tag{3.20}$$

$$\mathrm{d}V_t^a = -\nu^a V_t^a \,\mathrm{d}t + \sigma_a V_t^a \,\mathrm{d}W_t^a, \quad \mathrm{d}V_t^b = -\nu^b V_t^b \,\mathrm{d}t + \sigma_b V_t^b \,\mathrm{d}W_t^b. \tag{3.21}$$

In particular, $u_t|_{[-L,0]} \leq 0$, $u_t|_{[0,L]} \geq 0$ *and*

$$\nabla u_t(0+) = \frac{\pi}{L} V_t^a, \quad \nabla u_t(0-) = \frac{\pi}{L} V_t^b.$$

The L^1 normalization (3.18) allows us to interpret the variables in terms of order book volume and depth: $\int_0^L |u_t| = V_t^a$ (resp., $\int_{-L}^0 |u_t| = V_t^b$) represents the volume of sell (resp., buy) orders, while $\nabla u_t(0+)\theta = \frac{\theta\pi}{L} V_t^a$ (resp., $\nabla u_t(0-).\theta = \frac{\theta\pi}{L} V_t^b$) represents the depth at the top of the book. In this simple two-factor model, these two are proportional to each other: they may be decoupled by considering multifactor specifications involving higher-order eigenfunctions.

The drift parameter $-\nu^a$ (resp., $-\nu^b$) thus represents the net growth rate of decrease of the volume of sell (resp., buy) orders. As shown in (3.20), this net growth rate results from the superposition of several effects:

- submission/cancellation of limit sell (resp., buy) orders by directional sellers (resp., buyers) at rate α_a (resp., α_b); this may be interpreted as the "low-frequency" component of the order flow;
- replacement of limit orders by new ones closer to the mid-price, at rate $\frac{\beta_a^2}{4\eta_a}$ (resp., $\frac{\beta_b^2}{4\eta_b}$);
- cancellation of limit orders as the mid-price moves away (i.e., at distance $\pm L$ from the mid-price) at rate $\frac{\eta_a \pi^2}{L^2}$ (resp., $\frac{\eta_b \pi^2}{L^2}$).

In the case of a *balanced* order flow for which there is no systematic accumulation or depletion of limit orders away from the mid-price, these terms compensate each other and the volume of limit orders in any interval $[S_t + x_1, S_t + x_2]$ is a (local) martingale. The following result follows from the remarks in section 2.4.

Corollary 3.4 (balanced order flow). *The order book density u is a local martingale (in L^2) if and only if*

$$u_0(x) = V_0^b H_1^b(x) \mathbf{1}_{(-L,0)}(x) + V_0^a H_1^a(x) \mathbf{1}_{(0,L)}(x)$$

for some $V_0^b \geq 0, V_0^a \geq 0$ *and*

$$\alpha_a = \frac{\eta_a \pi^2}{L^2} + \frac{\beta_a^2}{4\eta_s}, \quad \alpha_b = \frac{\eta_b \pi^2}{L^2} + \frac{\beta_b^2}{4\eta_b}. \tag{3.22}$$

Remark 3.5 (balance between high- and low-frequency order flow). The balance condition (3.22) expresses a balance between the slow arrival of directional orders, represented by the terms α_a and α_b, and the fast replacement of orders inside the book, represented by the terms $\frac{\beta_a^2}{4\eta_b}$ and $\frac{\beta_a^2}{4\eta_b}$, and finally the cancellation of limit orders deep inside the book, at rate $\eta_a \pi^2 / L^2$.

This balance between order flow at various frequencies may be seen as a mathematical counterpart of the observations made by Kirilenko et al. (2017) on the nature of intraday order flow.

3.2. Shape of the order book. An implication of the above results is that the average profile of the order book is given, up to a constant, by the principal eigenfunctions H_1^a, H_1^b:

$$\mathbb{E}(u_t(x)) = \mathbb{E}(V_t^b)\ H_1^b(x) + \mathbb{E}(V_t^a)\ H_1^a(x). \tag{3.23}$$

Dropping the indices a, b, the normalized profile of the order book has the form

$$H_1(x) := c_1 e^{-\frac{\beta}{2\eta}x} \sin(\tfrac{\pi}{L}x), \qquad x \in [0, L],$$

where c_1 is such that $\int_0^L |H_1| = 1$:

$$\frac{1}{c_1} = \int_0^L e^{-\frac{\beta}{2\eta}x} \sin(\tfrac{\pi}{L}x)\ \mathrm{d}x = \frac{4\pi L\eta^2}{L^2\beta^2 + \pi^2 4\eta^2}\left(e^{-\frac{\beta}{2\eta}L} + 1\right).$$

Figure 3 shows this function for different values of β: H_1 has a unique maximum at

$$\hat{x} := \frac{L}{\pi} \arctan\left(\tfrac{2\eta\pi}{L\beta}\right). \tag{3.24}$$

The position of the maximum moves closer to the origin as β/η is increased. For $\beta = 0$ we have $\hat{x} = \frac{L}{2}$, and, on the other hand $\hat{x} \searrow 0$ as $\beta/\eta \to \infty$. Typically, the order book profile for liquid large tick securities a few ticks from the mid-price. Figure 4 shows the average order book profile for QQQ; similar results were found in Bouchaud, Farmer, and Lillo (2009) and Cont, Stoikov, and Talreja (2010). This suggests $\hat{x}$ is of the order of a few ticks, so we are interested in the parameter range for which β/η is large.

The value at the maximum is

$$\max_{x\in[0,L]} H_1(x) = \sqrt{\frac{\beta^2}{4\eta^2} + \frac{\pi^2}{L^2}} \exp\left(-\frac{\beta L}{2\eta\pi} \arctan\left(\frac{2\eta\pi}{L\beta}\right)\right)\left(e^{-\frac{\beta L}{2\eta}} + 1\right)^{-1}. \tag{3.25}$$

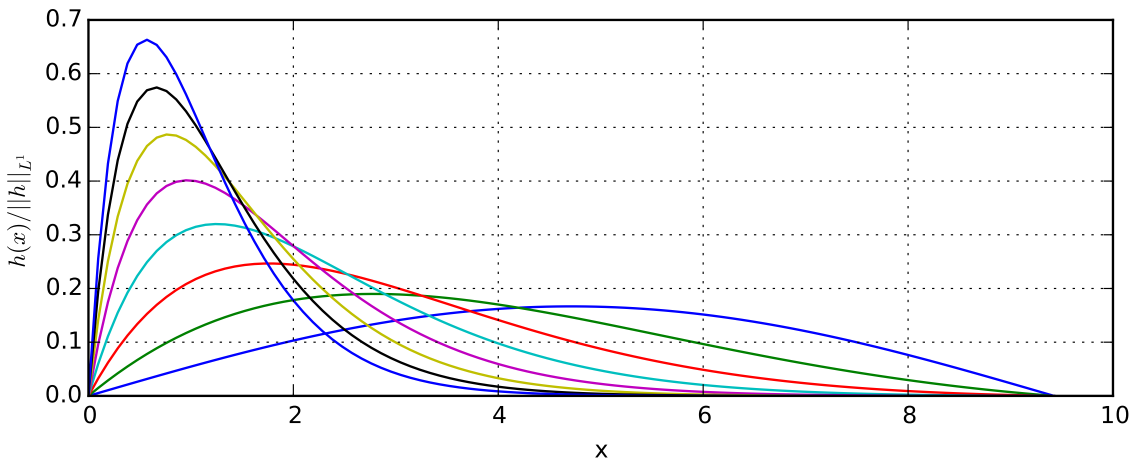

Figure 3. *Shape of the normalized principal eigenfunction H_1, which corresponds to the average profile of the normalized order book, for $L := 3\pi$, $\eta := 1$, and different values of $\beta \in \{0, 0.5, \ldots, 3.5\}$.*

which grows linearly as ${}^{\beta}\!/_{2\eta} \to \infty$, as shown in Figure 3, where we have plotted h, normalized by its L^1-norm, for various values of β with $L := 3\pi$ and $\eta = 1$.

The above results are valuable for calibrating the model parameters $\frac{\beta}{2\eta}$, α, and σ to reproduce the average profile (for each side) of the order book.

$\frac{\beta}{2\eta}$ can be estimated from the position $\hat{x}$ of the maximum using (3.24). Note that when L is large, then

$$\hat{x} \approx \frac{2\eta}{\beta}.$$

The height of this maximum gives a further constraint on parameters, using (3.25).

We will use this result for parameter estimation in section 3.6.

3.3. Dynamics of order book volume. As noted in Corollary 3.3, V_t^a and V_t^b may be identified as the volume of sell (resp., buy) limit orders: they follow (correlated) geometric Brownian motions:

$$V_t^a = \int_0^L |u_t(x)| \, \mathrm{d}x = V_0^a \exp\left(\sigma_a W_t^a - \nu_a t - \frac{\sigma_a^2 t}{2}\right),$$
$$V_t^b = \int_{-L}^0 |u_t(x)| \, \mathrm{d}x = V_0^b \exp\left(\sigma_b W_t^b - \nu_b t - \frac{\sigma_b^2 t}{2}\right),$$

where $[W^a, W^b]_t = \rho_{a,b} t$. The average volume of the order book $V_t = V_t^a + V_t^b$ satisfies

$$\mathbb{E}(V_t) = V_0 - \int_0^t V_a^0 \nu_a e^{-\nu_a s} - V_b^0 \nu_b e^{-\nu_b s} \, \mathrm{d}s = V_0 + V_a^0 e^{-\nu_a t} + V_b^0 e^{-\nu_b t}.$$

Intraday studies of order book volume show it to be stable away from the open and close. Here $\mathbb{E}V_t = V_a + V_b$ if and only if V is a martingale, i.e., $\nu_a = \nu_b = 0$.

3.4. Dynamics of price and market depth. Recall from the discussion in section 1.3 that the order book dynamics yield the price process

$$\mathrm{d}S_t = \theta \left(\frac{\mathrm{d}D_t^b}{D_t^b} - \frac{\mathrm{d}D_t^a}{D_t^a} \right),$$

where θ is an impact coefficient and D_t^b and D_t^a represent the depth at the top of the order book (Cont, Kukanov, and Stoikov, 2014):

$$D_t^a := \int_0^\delta u_t(x) \, \mathrm{d}x \approx \frac{1}{2}\delta^2 \nabla u_t(0+), \quad D_t^b := \int_{-\delta}^0 |u_t(x)| \, \mathrm{d}x \approx \frac{1}{2}\delta^2 \nabla u_t(0-). \tag{3.26}$$

Using the results in Corollary 3.3, we obtain the following price dynamics:

$$\mathrm{d}S_t = \theta \left(\frac{\mathrm{d}V_t^b}{V_t^b} - \frac{\mathrm{d}V_t^a}{V_t^a} \right), \tag{3.27}$$

where

$$\mathrm{d}V_t^b = -\nu_b V_t^b \, \mathrm{d}t + \sigma_b V_t^b \, \mathrm{d}W_t^b, \quad \mathrm{d}V_t^a = -\nu_a V_t^a \, \mathrm{d}t + \sigma_a V_t^a \, \mathrm{d}W_t^a. \tag{3.28}$$

The price dynamics can thus be written as

$$\begin{aligned} S_t &= S_0 - \theta t\,(\nu_b - \nu_a) + \theta\sigma_b W_t^b - \theta\sigma_a W_t^a \\ &= S_0 - \theta t\,(\nu_b - \nu_a) + \sigma_S B_t, \end{aligned}$$

where B is a Brownian motion and σ_S is the mid-price volatility, which may be expressed in terms of parameters describing the order flow:

$$\sigma_S := \theta\sqrt{\sigma_b^2 + \sigma_a^2 - 2\sigma_a\sigma_b\varrho_{a,b}}. \tag{3.29}$$

The implied price dynamics thus corresponds to the Bachelier model:

- ▷ The drift term $\nu_a - \nu_b$ depends only on the rate of relative increase of the bid/ask depth, not the actual depths D_t^b and D_t^a.
- ▷ The quadratic variation of the mid-price is $\sigma_S^2 t$ decreases with the correlation between the buy and sell order flow. This correlation, generated by market makers, reduces price volatility.

Remark 3.6. Replacing $\sigma_a W^a$ and $\sigma_b W^b$ by arbitrary semimartingales X^a and X^b with jumps bounded from below by -1 yields the following price dynamics:

$$S_t = S_0 - \theta t\,(\nu_b - \nu_a) + \theta(X_t^b - X_t^a). \tag{3.30}$$

In particular, this relation links price jumps to large changes ("jumps") in order flow imbalance:

$$\Delta S_t = \theta\ \Delta(X_t^b - X_t^a). \tag{3.31}$$

3.5. Absolute price coordinates: Stochastic moving boundary problem. The model above describes dynamics of the order book in relative price coordinates, i.e., as a function of the (scaled) distance x from the mid-price. The density of the limit order book parameterized by the (absolute) price level $p \in \mathbb{R}$ is given (in the case of linear scaling) by

$$v_t(p) = u_t(p - S_t), \qquad x \in \mathbb{R}, \tag{3.32}$$

where we extend u_t to $\mathbb{R}$ by setting $u_t(y) = 0$ for $y \in \mathbb{R} \setminus [-L, L]$. As observed in section 3.4, the mid-price dynamics is given by

$$\mathrm{d}S_t = -\theta(\nu_b - \nu_a)\ \mathrm{d}t + \theta\sigma_b\ \mathrm{d}W_t^b - \theta\sigma_a\ \mathrm{d}W_t^a. \tag{3.33}$$

The dynamics of v may then be described, via an application of the Itô–Wentzell formula, as the solution of a *stochastic moving boundary problem* (Mueller, 2018).

Theorem 3.7 (stochastic moving boundary problem). *The order book density $v_t(p)$, as a function of the price level p is a solution, in the sense of distributions, of the stochastic moving boundary problem*

$$\begin{aligned} \mathrm{d}v_t(p) = {}& \big[(\eta_a + \tfrac{1}{2}\sigma_s^2)\Delta v_t(p) \\ &+ (\nu_b - \nu_a + \beta_a - \theta(\varrho_{a,b}\sigma_b\sigma_a - \sigma_a^2)\nabla v_t(p) + \alpha_a v_t(p)\big]\ \mathrm{d}t \\ &+ (\sigma_a v_t(p) + \theta\sigma_a\nabla v_t(p))\ \mathrm{d}W_t^a - \theta\sigma_b\nabla v_t(p)\ \mathrm{d}W_t^b, \end{aligned} \tag{3.34}$$

for $p \in (S_t, S_t + L)$, *and*

$$\mathrm{d}v_t(p) = \big[(\eta_b + \tfrac{1}{2}\sigma_s^2)\Delta v_t(p) + (\nu_b - \nu_a - \beta_b - \theta(\sigma_b^2 - \varrho_{a,b}\sigma_b\sigma_a))\nabla v_t(x) + \alpha_b v_t(p)\big]\,\mathrm{d}t + \theta\sigma_a \nabla v_t(p)\,\mathrm{d}W_t^a + (\sigma_b v_t(p) - \theta\sigma_b\nabla v_t(p))\,\mathrm{d}W_t^b \tag{3.35}$$

for $x \in (S_t - L, S_t)$ *with the moving boundary conditions*

$$v_t(S_t) = 0, \qquad v_t(y) = 0, \qquad \forall y \in \mathbb{R} \setminus (S_t - L, S_t + L), \tag{3.36}$$

in the following sense: $(v_t)_{t\geq 0}$ *is a continuous* $L^2(\mathbb{R})$*-valued stochastic process and for all* $\varphi \in C_0^\infty(\mathbb{R})$ *and* $t \geq 0$,

$$\begin{aligned}\langle v_t, \varphi\rangle - \langle v_0, \varphi\rangle &= \int_0^t \langle m(x - S_t, \Delta v_r, \nabla v_r, v_r), \varphi\rangle\,\mathrm{d}r \\ &+ \frac{1}{2}\int_0^t (\nabla v_r(S_r-) - \nabla v_r(S_r+))\varphi(S_r) - \nabla v_r(S_r - L+)\varphi(S_r - L) \\ &+ \nabla v_r(S_r + L-)\varphi(S_r + L))\, d\langle S\rangle_r \\ &+ \int_0^t \langle \mathbf{1}_{(S,S_r+L)}\sigma_a v_r, \varphi\rangle\,\mathrm{d}W_r^a + \int_0^t \langle \mathbf{1}_{(S_r-L,S_r)}\sigma_b v_r, \varphi\rangle\,\mathrm{d}W_r^b \\ &+ \theta\sigma_a \int_0^t \langle \nabla v_r, \varphi\rangle\,\mathrm{d}W_r^a - \theta\sigma_b \int_0^t \langle \nabla v_r, \varphi\rangle\,\mathrm{d}W_r^b,\end{aligned} \tag{3.37}$$

where we denote, for $S \in \mathbb{R}$, $V \in H_0^1((-L, L) \setminus \{0\}) \cap H^2((-L, L) \setminus \{0\})$,

$$m(x, y'', y', y) = \begin{cases} (\eta_a + \frac{1}{2}\sigma_s^2)y'' \\ +(\nu_b - \nu_a + \beta_a - \theta(\varrho_{a,b}\sigma_b\sigma_a - \sigma_a^2))y' + \alpha_a y, & x \in (0, L), \\ (\eta_b + \theta\sigma_s^2)y'' \\ +(\nu_b - \nu_a - \beta_b - \theta(\sigma_b^2 - \varrho_{a,b}\sigma_b\sigma_a)))y' + \alpha_b y, & x \in (-L, 0), \\ 0 & \textit{else}, \end{cases}$$

for $x, y'', y', y \in \mathbb{R}$.

Remark 3.8. Note that (3.36) is a *stochastic boundary condition* at S_t.

The proof, given in Appendix A, is based on Krylov's extended Itô–Wentzell formula (Krylov, 2011, Theorem 1.1).

3.6. Parameter estimation. We now describe a method for estimating model parameters. We use time series of order books for NASDAQ stocks and ETFs from the LOBSTER database.

Given that we do not observe separately the various components of the order flow as in (3.1), we use the relations discussed in section 3.2 to calibrate the parameters σ, ν and the shape parameter

$$\gamma := \frac{\beta}{2\eta} \tag{3.38}$$

for each side of the order book. We set L to the largest value in our data set ($L := 1000$). Parameters may be calibrated either

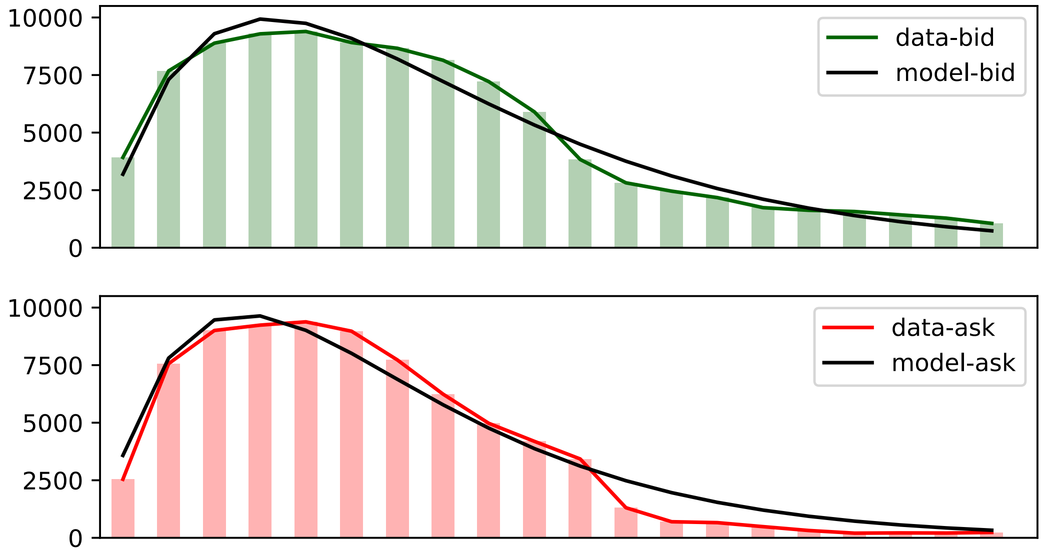

Figure 4. *Average profile of QQQ order book (first 20 levels), November 17, 2016 (top: bid; bottom: ask).*

(a) through a least squares fit of (3.23) to the average order book profile, or
(b) to reproduce the position $\hat{x}$ and height of the maximum of the order book profile.

Remark 3.9. The estimator based on the maximum position of the peak is fast in computation but the fixed price level grid in the data restricts the set possible values for estimation of γ. In particular, the estimator is sensitive to the location of the maximum (i.e., the mode of the order book profile).

We show results for a set of NASDAQ stocks and ETFs. Figure 4 shows how the model reproduces the average book profile for QQQ at NASDAQ on November 17, 2017. In Figure 5 we see the coefficient γ estimated across various 30-min windows during the trading day. The one-factor model based on the principal eigenfunction yields a reasonable approximation for the average order book profile, which justifies our assumptions on the dynamics in section 1.2.

For low-price/large tick stocks, the average order book profiles may differ from the exponential-sine shape. For such stocks, we use the nonlinear scaling described in section 1.1, leading to an average order book profile:

$$U(p) = V\exp(-\gamma((p - S_t)/\delta)^a)\sin\left(((p - S_t)/\delta)^a\pi/L\right), \tag{3.39}$$

where S_t is the best price. Figure 6 shows such a nonlinear fit for the average order book profile of SIRI.

4. Mean-reverting models.

4.1. A class of models with mean reversion. We now return to the full model (1.2) with nonzero source terms $f^a(x), f^b(x)$ representing the rate of arrival of new limit orders at a distance x from the best price:

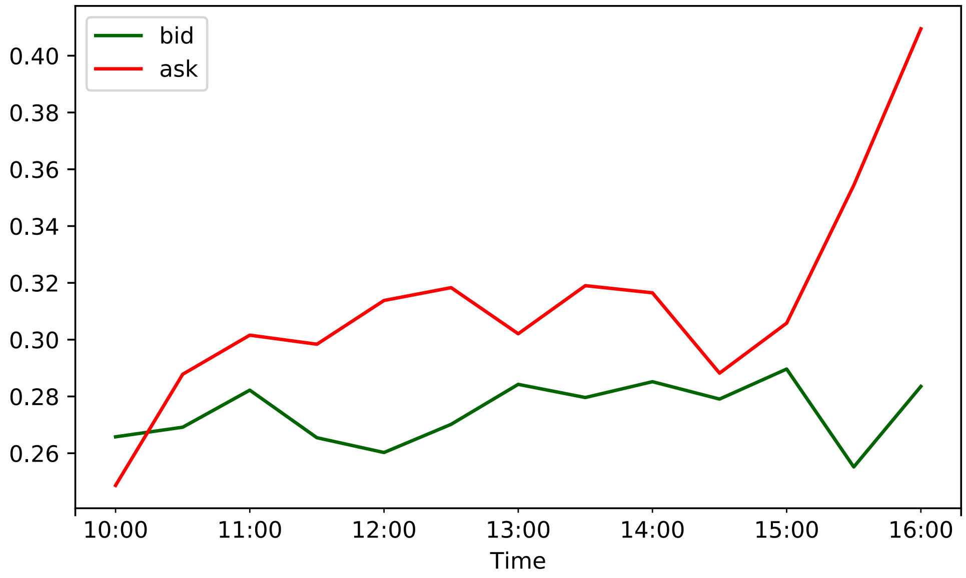

Figure 5. *Values of parameter γ_a, γ_b estimated from 30 min. average profile of QQQ order book (first 20 levels), November 17, 2016.*

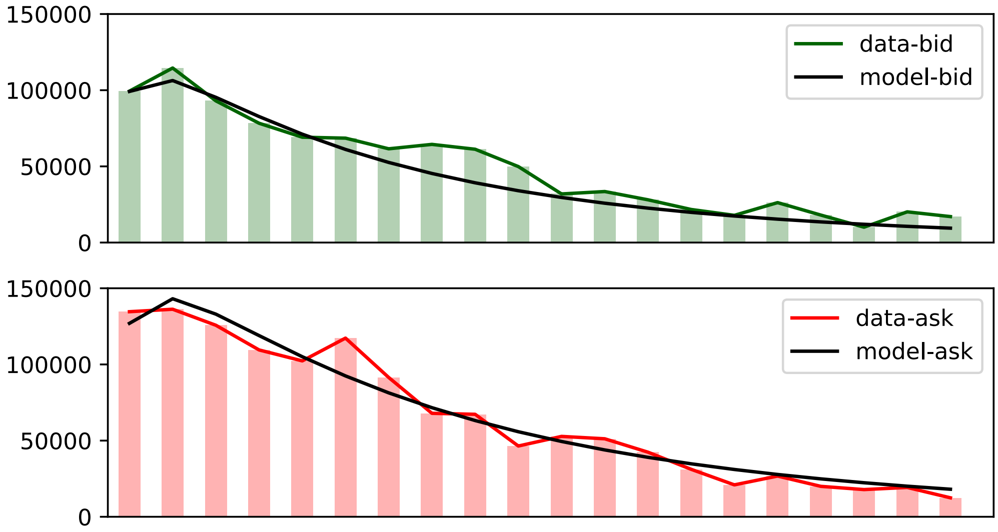

Figure 6. *Average profile of SIRI order book, first 20 levels, November 17, 2016 (top: bid; bottom: ask), $\gamma_b = 0.95$, $\gamma_a = 0.86$, $a_b = 0.52$, $a_a = 0.56$.*

$$
\begin{aligned}
&\mathrm{d}u_t(x) = [\eta_a \Delta u_t(x) + \beta_a \nabla u_t(x) + \alpha_a u_t(x) + f^a(x)]\ \mathrm{d}t + \sigma_a u_t(x)\ \mathrm{d}W_t^a, \quad x \in (0, L),\\
&\mathrm{d}u_t(x) = \left[\eta_b \Delta u_t(x) - \beta_b \nabla u_t(x) + \alpha_b u_t(x) + f^b(x)\right]\ \mathrm{d}t + \sigma_b u_t(x)\ \mathrm{d}W_t^b, \quad x \in (-L, 0),\\
&u_t(0+) = u_t(0-) = 0, \qquad u_t(-L) = u_t(L) = 0, \quad t > 0,
\end{aligned}
$$

with the sign condition

$$u_t(x) \le 0, \quad x \in (-L, 0), \quad \text{and} \quad u_t(x) \ge 0, \quad x \in (0, L),\ t \ge 0,$$

where, as above η_a, η_b, σ_a, $\sigma_b > 0$, β_a, $\beta_b \ge 0$, α_a, $\alpha_b \in \mathbb{R}$ are constants and $u_0 \in L^2((-L, L))$. As above, we denote $u_0^b := u_0|_{[-L,0]}$ and $u_0^a := u_0|_{[0,L]}$. We will show that, when α_a and α_b are negative and $f^a(x) > 0$, $f^b(-x) < 0$ for all $x \in (0, L)$, this class of models leads to mean-reverting dynamics for the order book profile, consistent with the observation that intraday dynamics of order book volume and queue size over intermediate time scales (hours, day) typically exhibit mean reversion rather than a trend.

Projecting the equation on the eigenfunctions h_k^a, h_k^b, as in section 3, we see that, due to the fast increase in the eigenvalues (3.4), solutions starting from a generic initial condition may be approximated by their projection on the principal eigenfunctions h_1^a, h_1^b (we will justify this below in Proposition 4.2) and the main contribution of heterogeneous order arrivals arises from the projection of f^a (resp., f^b) on h_1^a (resp., h_1^b).

This motivates the following specfication, which leads to a tractable class of models:

$$f^a(x) := \bar{V}_a\ H_1^a(x), \qquad f^b(x) := \bar{V}_b\ H_1^b(x), \quad \bar{V}_a > 0, \quad \bar{V}_b > 0. \tag{4.1}$$

Theorem 2.10 then gives explicit solutions to (1.2). Recall the notation (3.10) and (3.9) and define V_t^b and V_t^a by

$$\mathrm{d}V_t^a = \left(\bar{V}_a - \nu_a V_t^a\right) \mathrm{d}t + \sigma_a V_t^a\ \mathrm{d}W_t^a,\ \mathrm{d}V_t^b = \left(\bar{V}_b - \nu_b V_t^b\right) \mathrm{d}t + \sigma_b V_t^b\ \mathrm{d}W_t^b, \tag{4.2}$$

where $\nu_i := \frac{\eta_i \pi^2}{L^2} + \frac{\beta_i^2}{4\eta_i} - \alpha_i$, $i \in \{a, b\}$. The solution of the SPDE may then be obtained as follows.

Proposition 4.1. (i) *The unique L^2-continuous solution of* (1.2)–(4.1) *for a general initial condition u_0 is given by*

$$u_t(x) = \begin{cases} V_t^b H_1^b(x) + \mathcal{E}_t(\sigma_b W^b) \sum_{k=1}^\infty e^{-\nu_k^b t} \left\langle u_0^b - V_0^b H_1^b, h_k^b \right\rangle_{(-\frac{\beta_b}{2\eta_b})} h_k^b(x), & x \in (-L, 0), \\ V_t^a H_1^a(x) + \mathcal{E}_t(\sigma_a W^a) \sum_{k=1}^\infty e^{-\nu_k^a t} \left\langle u_0^a - V_0^a H_1^a, h_k^a \right\rangle_{\frac{\beta_a}{2\eta_a}} h_k^a(x), & x \in (0, L), \\ 0, \quad x \notin (-L, 0) \cup (0, L). \end{cases}$$

(ii) *For an initial condition of the form*

$$u_0(x) = V_0^a H_1^a(x) \mathbf{1}_{[0,L]} + V_0^b H_1^b(x) \mathbf{1}_{[-L,0]}$$

the unique L^2-continuous solution of (1.2)–(4.1) *is given by*

$$u_t(x) = \left(V_t^a H_1^a(x) \mathbf{1}_{(0,L)}(x) + V_t^b H_1^b(x) \mathbf{1}_{(-L,0)}(x)\right), \quad x \in [-L, L]. \tag{4.3}$$

Proof. We obtain the general solution of the linear homogeneous equation from Proposition 3.2. The series representation of u results from the spectral decomposition, Proposition 3.1, and Theorem 2.10. ∎

4.2. Long time asymptotics and stationary solutions. In order to derive properties of the 'average' order book profile, we now examine whether the order book profile u_t has an ergodic behavior and describe stationary solutions. The following result describes the long-term dynamics and shows that this dynamics is well approximated by projecting the initial condition on the principal eigenfunctions as done in (4.3):

Proposition 4.2. *Let u_t be the unique solution of (1.2)–(4.1) for a general initial condition $u_0 \in L^2(-L, L)$ and define*

$$\check{u}_t(x) := V_t^b H_1^b(x)\mathbf{1}_{(-L,0)}(x) + V_t^a H_1^a(x)\mathbf{1}_{(0,L)}(x), \qquad t > 0. \tag{4.4}$$

If $\nu_1^b > 0$ and $\nu_1^a > 0$, then we have as follows:

(i) *The long-term dynamics of the order book is well approximated by the dynamics (4.4) projected along the principal eigenfunctions:*

$$\mathbb{P}\left(\lim_{t\to\infty} \|u_t - \check{u}_t\|_\infty = 0 \quad\right) = 1. \tag{4.5}$$

(ii) *u_t has a unique stationary distribution and*

$$u_t(x) \underset{t\to\infty}{\Longrightarrow} f^b(x)Z^b, \quad x < 0, \qquad u_t(x) \underset{t\to\infty}{\Longrightarrow} f^a(x)Z^a, \quad x > 0, \tag{4.6}$$

where f^a, f^b are given by (4.1) and Z^a (resp., Z^b) is an inverse Gamma random variable with shape parameter $1 + 2\frac{\nu_a}{\sigma_a^2}$ (resp., $1 + 2\frac{\nu_b}{\sigma_b^2}$) and scale parameter $\frac{\sigma_a^2}{2V_a}$ (resp., $\frac{\sigma_b^2}{2V_b}$).

(iii) *If furthermore $\nu_1^b > \frac{\sigma_b^2}{2}$ and $\nu_1^a > \frac{\sigma_a^2}{2}$, then*

$$\lim_{t\to\infty} \mathbb{E}\left[\|u_t - \check{u}_t\|^2_{L^2(-L,L)}\right] = 0. \tag{4.7}$$

Proof. For $t_0 > 0$, let

$$K_{t_0} := \sqrt{\sum_{k=1}^\infty e^{-2(\nu_k^a - \nu_1^a)t_0}} < \infty.$$

Denote $u_t^\circ(.; h)$ the unique solution of the linear homogeneous equation (3.1) for an initial condition h. Recall from Theorem 2.10 that

$$u_t(x) - \check{u}_t(x) = u_t^\circ(x; u_0 - \check{u}_0).$$

It suffices now to prove the results for the ask side and note that the calculations will be analogous for the bid side. Using the representation of u_t° from Proposition 3.2 we get for all $t > t_0$ and all $h \in L^2(0, L)$,

$$\begin{aligned}\left\|u_t^\circ(.; h)|_{(0,L)}\right\|_\infty &\le e^{-\nu_1^a t}\mathcal{E}_t(\sigma_a W^a)\sqrt{\sum_{k=1}^\infty e^{-2(\nu_k^a-\nu_1^a)t_0}}\sqrt{\sum_{k=1}^\infty \left|\langle h, h_k^a\rangle_{\frac{\beta_a}{2\eta_a}}\right|^2}\\ &= K_{t_0}\,\|h\|_{\frac{\beta_a}{2\eta_a}} \exp\left(\sigma_a W_t^a - \left(\nu_1^a + \frac{\sigma_a^2}{2}\right)t\right),\end{aligned}$$

which, as $t \to \infty$, converges to 0 provided that $\nu_1^a > 0$. This proves (i).

To show (iii), a similar calculation but using the orthogonality of the decomposition in Proposition 3.2 yields

$$\begin{aligned}\mathbb{E}\left[\left\|u_t^{\circ}(.;h)|_{(0,L)}\right\|^2_{\frac{\beta_a}{2\eta_a}}\right] &= \sum_{k=1}^{\infty} e^{-2\nu_k^a t}\left|\langle h, h_k^a\rangle_{\frac{\beta_a}{2\eta_a}}\right|^2 \mathbb{E}\left[|\mathcal{E}_t(\sigma_a W^a)|^2\right]\\ &\le e^{-2\nu_1^a t}\|h\|^2_{\frac{\beta_a}{2\eta_a}}\mathbb{E}\left[\exp\left(2\sigma_a W_t^a - \sigma_a^2 t\right)\right]\\ &= e^{(-2\nu_1^a+\sigma_a^2)t}\|h\|^2_{\frac{\beta_a}{2\eta_a}}.\end{aligned}$$

If $\sigma_a^2 < 2\nu_1^a$, then this converges to 0 as $t \to \infty$. Since $\|.\|_{\frac{\beta_a}{2\eta_a}}$ defines an equivalent norm on $L^2(0,L)$, this finishes the proof of (iii).

Assertion (ii) follows from Proposition 2.13. Indeed, recall that $V^i, i \in \{a,b\}$, are ergodic processes whose unique invariant distribution is given by an inverse Gamma distribution with shape parameter $1+\frac{2\nu_i}{\sigma_i^2}$ and scale parameters $\frac{\sigma_i^2}{(\bar{V}_i)^2}$, $i \in \{a, b\}$. Denote by Z^b and Z^a random variables with these distribution For any $x \in [-L,L]$, we have the convergence in distribution

$$\check{u}_t|_{(-L,0)} \Longrightarrow Z^b\ f_1^b(.), \qquad \check{u}_t|_{(0,L)} \Longrightarrow Z^a\ f^a(.). \tag{4.8}$$

Since almost sure convergence yields convergence in distribution, by part (i) this yields that (4.8) holds also for u_t with arbitrary initial data $u_0 \in L^2(-L,L)$. ■

4.3. Dynamics of order book volume. Consider now the "projected" dynamics as in the setting of Proposition 4.1(ii). The dynamics of the order book volume V_t is then given by

$$V_t := \int_{-L}^{L} |u_t(x)|\ \mathrm{d}x = V_t^b + V_t^a, \qquad t \ge 0, \tag{4.9}$$

where V^b and V^a, defined in (4.2), represent the volume of buy (resp., sell) orders in the order book.

Since $[W^a, W^b]_t = \varrho_{a,b} t$ we can write

$$W^a =: W, \qquad W^b := \varrho_{a,b} W + \sqrt{1-\varrho_{a,b}^2}\widehat{W}$$

for some Brownian motion $\widehat{W}$, independent of W. Then,

$$\begin{aligned}\mathrm{d}V_t &= (\bar{V}_a + \bar{V}_b - (\nu_a V_t^a + \nu_b V_t^b)\ \mathrm{d}t\\ &\quad + \left(\sigma_a V_t^a + \varrho_{a,b}\sigma_b V_t^b\right)\mathrm{d}W_t + \sqrt{1-\varrho_{a,b}^2}\sigma_b V_t^b\ \mathrm{d}\widehat{W}_t.\end{aligned} \tag{4.10}$$

In particular, the quadratic variation ("realized variance") of the order book volume is given by

$$\mathrm{d}\langle V\rangle_t = \left(\sigma_a^2 (V_t^a)^2 + 2\varrho_{a,b}\sigma_b\sigma_a V_t^a V_t^b + \sigma_b^2 (V_t^b)^2\right)\mathrm{d}t. \tag{4.11}$$

For the symmetric and perfectly correlated case, V is itself a reciprocal gamma diffusion.

Corollary 4.3. *Assume the setting of Proposition 4.1(ii) and, in addition, that* $\nu_a = \nu_b =: \nu$, $\sigma_a = \sigma_b =: \sigma$, *and* $\varrho_{a,b} = 1$. *Then,* V *is the unique solution of*

$$\mathrm{d}V_t = \big((\bar V_b + \bar V_a) - \nu V_t\big)\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t \tag{4.12}$$

with $V_0 = V_0^b + V_0^a$.

In all cases, we get from (2.22) that for $i \in \{a,b\}$, $t \geq 0$,

$$\mathbb{E}V_t^i = \left(V_0^i - \frac{\nu_i}{\bar V_i}\right) e^{-\nu_i t} + \frac{\nu_i}{\bar V_i} \tag{4.13}$$

and

$$\mathbb{E}V_t^i = \left(V_0^b - \frac{\bar V_b}{\nu_b}\right) e^{-\nu_b t} + \left(V_0^a - \frac{\bar V_a}{\nu_a}\right) e^{-\nu_a t} + \frac{\bar V_b}{\nu_b} + \frac{\bar V_a}{\nu_a}. \tag{4.14}$$

4.4. Joint dynamics of mid-price and market depth. We now consider the mid-price and market depths dynamics in the situation of Proposition 4.1(ii). As discussed in sections 1.3 and 3.4 for the linear homogeneous models, the dynamics of the mid-price is given by

$$\mathrm{d}S_t = \theta\left(\frac{\mathrm{d}D_t^b}{D_t^b} - \frac{\mathrm{d}D_t^a}{D_t^a}\right),$$

where θ is an impact coefficient, while the bid/ask depths follow

$$\begin{aligned}
D_t^a &:= \int_0^\delta u_t(x)\,\mathrm{d}x \approx \frac{1}{2}\delta^2 \nabla u_t(0+) = \frac{\pi}{2L}\delta^2 V_t^a,\\
D_t^b &:= -\int_{-\delta}^0 u_t(x)\,\mathrm{d}x \approx \frac{1}{2}\delta^2 \nabla u_t(0-) = \frac{\pi}{2L}\delta^2 V_t^a.
\end{aligned}$$

Thus, the dynamics of the market depths are given by

$$\begin{aligned}
\mathrm{d}D_t^b &= \nu_b\left(\overline{D}_b - D_t^b\right)\mathrm{d}t + \sigma_b D_t^b\,\mathrm{d}W_t^b,\\
\mathrm{d}D_t^a &= \nu_a\left(\overline{D}_a - D_t^a\right)\mathrm{d}t + \sigma_a D_t^a\,\mathrm{d}W_t^a
\end{aligned}$$

for some mean reversion levels $\overline{D}_b, \overline{D}_a > 0$. We thus obtain the joint dynamics of price and market depth:

$$\begin{aligned}
\mathrm{d}\begin{pmatrix} D_t^b \\ D_t^a \\ S_t \end{pmatrix} &= \begin{pmatrix} \nu_b(\overline{D}_b - D_t^b) \\ \nu_a(\overline{D}_a - D_t^a) \\ \theta\left(\frac{\nu_b \overline{D}_b}{D_t^b} - \frac{\nu_a \overline{D}_a}{D_t^a} - (\nu_b - \nu_a)\right) \end{pmatrix}\mathrm{d}t\\
&\quad + \begin{pmatrix} \sigma_b D_t^b & 0 \\ \varrho_{a,b}\sigma_a D_t^a & \sqrt{1-\varrho_{a,b}^2}\,\sigma_a D_t^a \\ \theta\left(\sigma_b - \varrho_{a,b}\sigma_a\right) & -\theta\sqrt{1-\varrho_{a,b}^2}\,\sigma_a \end{pmatrix}\mathrm{d}\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix},
\end{aligned} \tag{4.15}$$

where W^1 and W^2 are independent Brownian motions. The mid-price itself has quadratic variation $\langle S \rangle_t = \sigma_S^2 t$, where

$$\sigma_S := \theta \sqrt{\sigma_b^2 + \sigma_a^2 - 2\sigma_a \sigma_b \varrho_{a,b}}. \tag{4.16}$$

Over a small time interval Δt,

$$\begin{aligned} S_{\Delta t} &= S_0 + \theta \int_0^{\Delta t} \frac{\nu_b(\overline{D}_b - D_s^b)}{D_s^b} - \frac{\nu_a(\overline{D}_a - D_s^a)}{2D^a(s)} \, \mathrm{d}s + \theta\sigma_b W_{\Delta t}^b - \theta\sigma_a W_{\Delta t}^a \\ &\approx S_0 + \Delta t \frac{\theta}{2} \left(\frac{\nu_b(\overline{D}_b - D_0^b)}{D_0^b} - \frac{\nu_a(\overline{D}_a - D_0^a)}{D_0^a} \right) + \sigma_S \sqrt{\Delta t} \, N_{0,1}, \end{aligned}$$

where $N_{0,1}$ is a standard Gaussian variable. In particular the conditional probability of an upward mid-price move of size y is given by

$$\mathbb{P}\left[S_{\Delta t} \geq S_0 + y\right] \simeq N\left(\frac{\theta\sqrt{\Delta t}}{\sigma_S} \left(\frac{\nu_b(\overline{D}_b - D_0^b)}{D_0^b} - \frac{\nu_a(\overline{D}_a - D_0^a)}{D_0^a} \right) - \frac{y}{\sigma_S\sqrt{\Delta t}} \right), \tag{4.17}$$

where N denotes the cumulative distribution function of the standard normal distribution.

Remark 4.4. Using (2.22), the expected order flow over a small time interval $[0,t]$ on each side of the book is given by for $\star \in \{a, b\}$,

$$\mathbb{E}\left[D_t^\star - D_0^\star\right] = t\nu_\star(\overline{D}_\star - D_0^\star) + o(t). \tag{4.18}$$

Remark 4.5 (mean-reverting order book imbalance). The imbalance between buy and sell depth is a frequently used indicator for predicting short term price moves (Cartea, Donnelly, and Jaimungal, 2018; Cont and de Larrard, 2013; Lipton, Pesavento, and Sotiropoulos, 2014). In this model, the depth imbalance has the following dynamics:

$$\mathrm{d}(D_t^b - D_t^a) = \left(\nu^b \overline{D}^b - \nu^a \overline{D}^a - (\nu^b D_t^b - \nu^a D_t^a) \right) \mathrm{d}t + \sigma_b D_t^b \, \mathrm{d}W_t^b - \sigma_a D_t^a \, \mathrm{d}W_t^a.$$

In the symmetric case, when $\overline{D} = \overline{D}_a = \overline{D}_b$, $\nu = \nu_a = \nu_b$, (4.17) becomes

$$N\left(\frac{\nu\overline{D}\theta\sqrt{t}}{\sigma_S} \frac{(D_0^a - D_0^b)}{D_0^a D_0^b} - \frac{y}{\sigma_S\sqrt{t}} \right). \tag{4.19}$$

This quantity is *decreasing* in the depth imbalance $D_0^b - D_0^a$: this is a consequence of the mean reversion in order book depth. In the symmetric case

$$\mathrm{d}(D_t^b - D_t^a) = -\nu \left(D_t^b - D_t^a \right) \mathrm{d}t + \sigma_b D_t^b \, \mathrm{d}W_t^b - \sigma_a D_t^a \, \mathrm{d}W_t^a, \tag{4.20}$$

so the model reproduces the empirical observation that order book imbalance is mean reverting (Cartea, Donnelly, and Jaimungal, 2018).

Note that the model predicts mean reversion of market depths on the scale of $1/\nu$ which corresponds to seconds for the ETFs QQQ and SPY and around 10 seconds for large tick stocks such as MSFT and INTC (see Table 1). For time scales smaller than $1/\nu$, the direction of price moves is highly correlated with order flow imbalance, as shown in empirical studies of equity markets (Cont, Kukanov, and Stoikov, 2014).

4.5. Parameter estimation. We now discuss estimation of model parameters from a discrete set of observations $(V_n^a, V_n^b)_{n=0,\dots,N}$ of the bid/ask volumes V_t^a, V_t^b on a uniform time grid $\{k\Delta t : k = 0, \dots, N\}$. Let us rewrite the dynamics of V_t^a and V_t^b in the form of reciprocal Gamma diffusions:

$$\mathrm{d}V_t^\star = \nu_\star\left(\overline{D}_\star - V_0^\star\right) + \sqrt{2\frac{\nu_\star}{c_\star}(V_t^\star)^2}\,\mathrm{d}W_t^\star, \quad t \geq 0, \quad V_0^\star \in (0,\infty), \star \in \{a,b\} \tag{4.21}$$

with $\nu_\star$, $\overline{D}_\star$, $c_\star > 0$. We use method of moments estimators as in Leonenko and Šuvak (2010) for $\overline{D}_\star$ and $c_\star$ and a martingale estimation function (Bibby and Sørensen, 1995) for the autocorrelation parameters $\nu_\star$, $\star \in \{a,b\}$: we define

$$\widehat{\overline{D}}_\star := \frac{1}{N}\sum_{k=1}^N \hat{V}_k \quad \text{and} \quad \hat{c}_\star := \frac{\sum_{n=1}^N (\hat{V}_n)^2}{\sum_{n=1}^N (\hat{V}_n)^2 - \widehat{\overline{D}}_\star^2} = 1 + \frac{\widehat{\overline{D}}_\star^{\,2}}{\sum_{n=1}^N |\hat{V}_n|^2 - \widehat{\overline{D}}_\star^{\,2}}.$$

Combining Proposition 2.13 and Remark 2.16 with Theorem 6.3 of Leonenko and Šuvak (2010) we obtain that if $\overline{D}_\star > 0$ and $c_\star > 5$, then $V^\star$ has finite fourth moment and the estimators are consistent and asymptotically normal.

For the autocorrelation parameters ν_a and ν_b we use the martingale estimation function (Bibby and Sørensen, 1995, section 2)

$$G_\star(\nu; \overline{D}, c) := \frac{c}{\nu}\sum_{n=1}^N \frac{(\overline{D}_\star - \hat{V}_{n-1}^\star)}{(\hat{V}_{n-1})^2}\left(\hat{V}_n - F(\hat{V}_{n-1}; \nu, \overline{D})\right), \tag{4.22}$$

where

$$F(z; \nu, \overline{D}) := (z - \overline{D})e^{-\nu\Delta t} + \overline{D}. \tag{4.23}$$

Given $\overline{D}_\star$, this yields the estimators

$$\hat{\nu}_\star := \frac{1}{\Delta t}\log\left(-\frac{\sum_{n=1}^N \frac{(\overline{D}_\star - \hat{V}_{n-1})^2}{(\hat{V}_{n-1})^2}}{\sum_{n=1}^N \frac{(\overline{D}_\star - \hat{V}_{n-1})}{(\hat{V}_{n-1})^2}(V_n - \overline{D}_\star)}\right), \qquad \star \in \{a,b\}. \tag{4.24}$$

Convergence of this estimator is discussed in Bibby and Sørensen (1995, Theorem 3.2).

We apply these estimators to high-frequency limit order book time series for NASDAQ stocks and ETFs, obtained from the LOBSTER database, arranged into equally spaced observations over time intervals of size $\Delta t = 10ms$ and $\mathrm{d}t = 50ms$. For each observation we use as market depth the average volume of order in the first two price levels, respectively on bid and ask sides.[3] Below we show sample results for ETFs (SPY and QQQ) and liquid stocks (MSFT and INTC).

Table 1 shows estimated parameter values across different days for INTC, MSFT, QQQ, and SPY. We observe negative values of correlation $\varrho_{a,b}$ across bid and ask order flows, which is consistent with observations in Carmona and Webster (2019).

[3]The source code for the implementation is available online (Cont and Mueller, 2018).

Table 1
Averaged estimators for model parameters; ν and σ are given per second.

Ticker	Date	μ_b	μ_a	ν_b	ν_a	σ_b	σ_a	$\varrho_{a,b}$
INTC	2016-11-15	5179.0	5641.7	0.151	0.156	0.133	0.134	-0.077
	2016-11-16	5565.0	5672.5	0.082	0.118	0.111	0.124	-0.070
	2016-11-17	5776.5	7363.2	0.144	0.109	0.118	0.116	-0.019
MSFT	2016-11-15	3035.6	3855.9	0.522	0.426	0.292	0.292	-0.092
	2016-11-16	2839.9	3562.1	0.409	0.395	0.239	0.240	-0.071
	2016-11-17	4149.0	5762.5	0.300	0.239	0.202	0.208	-0.146
QQQ	2016-11-15	4686.9	5489.2	2.467	1.972	0.724	0.639	-0.177
	2016-11-16	4801.0	5142.6	2.041	1.845	0.632	0.677	-0.177
	2016-11-17	6414.0	6226.4	1.428	1.281	0.510	0.506	-0.224
SPY	2016-11-15	3903.4	4877.9	1.949	1.689	0.737	0.666	-0.176
	2016-11-16	3773.4	4486.4	1.324	1.763	0.578	0.657	-0.156
	2016-11-17	3693.0	4115.4	1.355	1.405	0.597	0.543	-0.181

Figures 7 and 8 show intraday variation of estimators for ν_a, ν_b, σ_a, σ_b, and $\varrho_{a,b}$ computed over 15-minute windows.

There are various estimators for intraday price volatility in this model, which allow us to test the model. Recall that in (4.16) we expressed price volatility in terms of the parameters describing the order flow:

$$\hat{\sigma}_S := \theta\sqrt{\sigma_b^2 + \sigma_a^2 - 2\sigma_b\sigma_a\varrho_{a,b}}, \tag{4.25}$$

where θ is the impact coefficient. We call this the RV estimator.

Another estimator is obtained by first estimating σ_b and σ_a using the martingale estimation function (4.22) then computing the price volatility using (4.25). We label this the RCG estimator.

Finally, one can compute the realized variance of the price over a 30 minute time window using price changes over 10 ms intervals. Comparing these different estimators is a qualitative test of the model.

Figure 9 compares these estimators, computed over 30 minute time windows: we observe that the model-based estimators are of the same order and closely track the intraday realized price volatility, which shows that the model captures correctly the qualitative relation between order flow and volatility.

Appendix A. Dynamics in absolute price coordinates. We now discuss in more detail the generalized Itô–Wentzell formula for distribution-valued processes, which is used in section 3.5 to derive the dynamics of the (noncentered) order book density $v_t(p)$. Let $C_0^\infty := C_0^\infty(\mathbb{R})$ be the space of smooth compactly supported functions on $\mathbb{R}$, $\mathbb{D}$ its dual, the space of generalized functions. We denote by $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ the first two derivatives in the sense of distributions and by $\langle .,. \rangle$ the duality product on $\mathbb{D} \times C_0^\infty$.

A $\mathbb{D}$-valued stochastic process $u = (u_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called $\mathbb{F}$-*predictable* if for all $\phi \in C_0^\infty(\mathbb{R})$ the real-valued process $(\langle u_t, \phi\rangle)_{t\geq 0}$ is predictable.

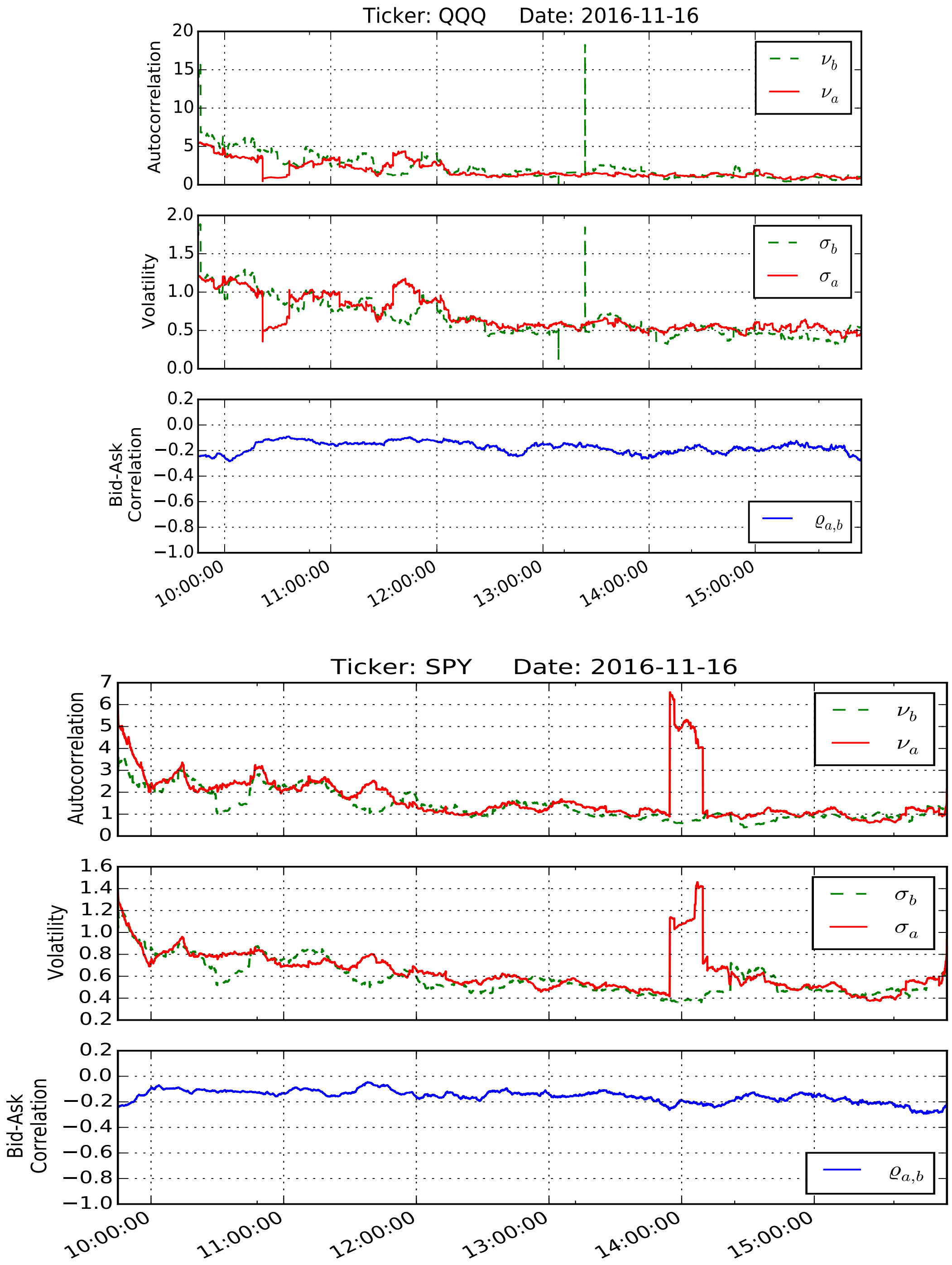

Figure 7. *Autocorrelation ($\nu_{a/b}$), standard deviation ($\sigma_{a/b}$), and bid-ask correlation ($\varrho_{a,b}$) of order book depth in first two levels for two liquid ETFs (QQQ and SPY).*

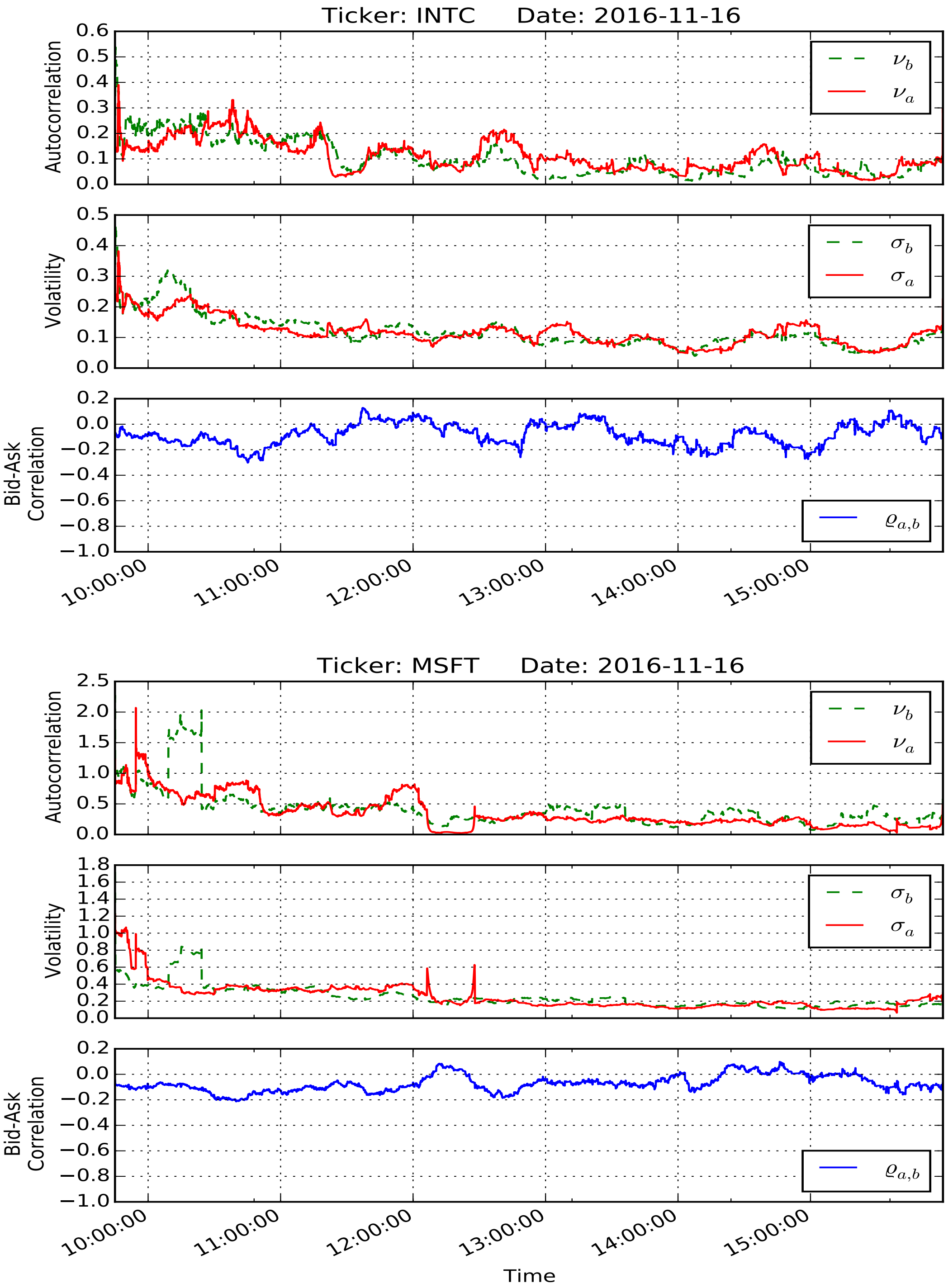

Figure 8. *Autorcorrelation ($\nu_{a/b}$), standard deviation ($\sigma_{a/b}$), and bid-ask correlation ($\varrho_{a,b}$) of order book depth in first two levels for two liquid stocks (INTC and MSFT).*

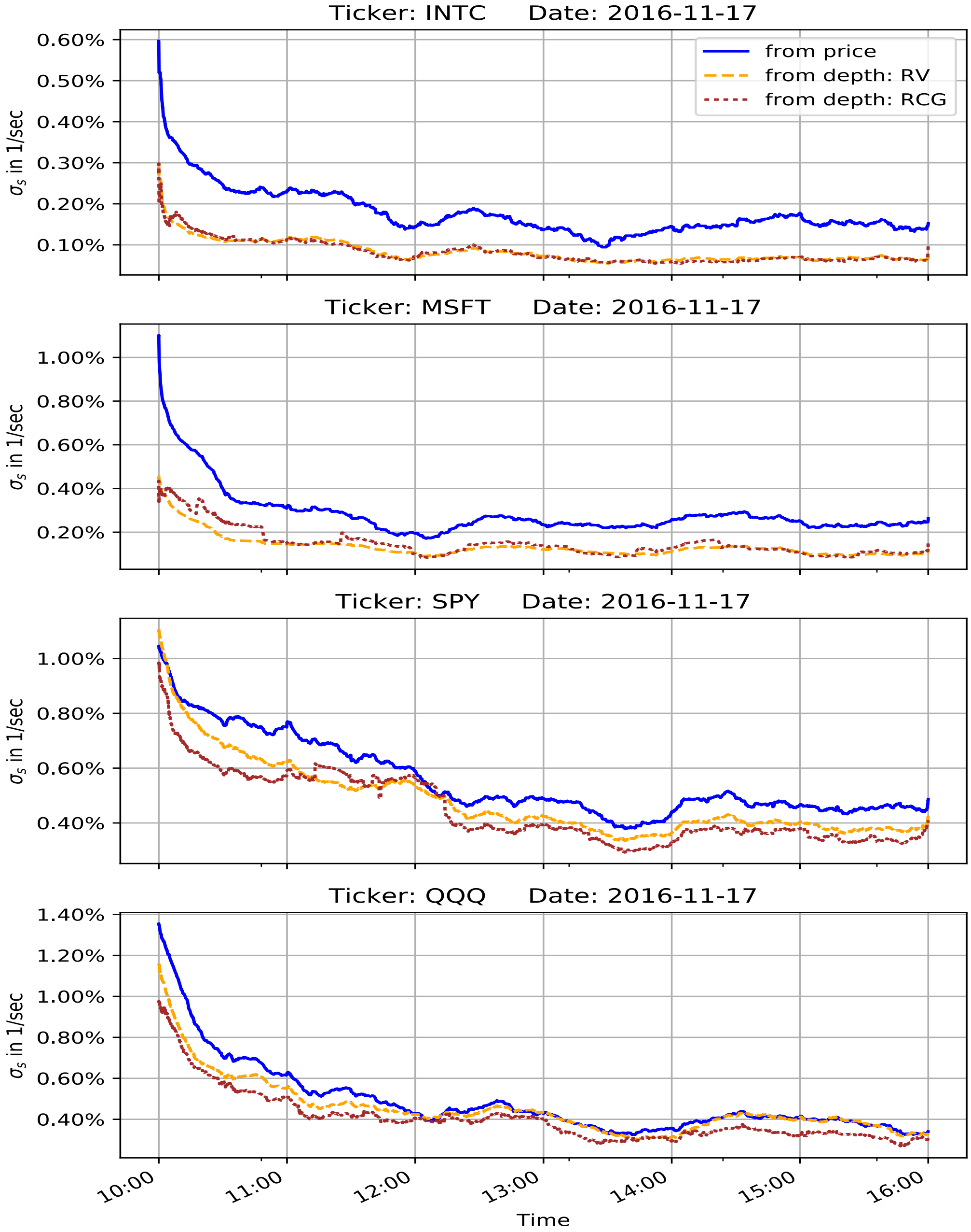

Figure 9. *Comparison of various estimators for intraday price volatility σ_S: standard deviation of price changes (blue), estimator based on realized variance/covariance of bid/ask depth (red), and estimator based on martingale estimation function (orange).*

Let $N \in \mathbb{N}$ and $(b_t)_{t\geq 0}$ and $(c_t^k)_{t\geq 0}$, $k \in \{1,\dots,N\}$, be predictable $\mathbb{D}$-valued processes. We assume that for all T, $R \in (0,\infty)$ and all $\phi \in C_0^\infty(\mathbb{R})$, almost surely

$$\text{(A.1)} \qquad \int_0^T \sup_{|x|\leq R} |\langle b_t, \phi(.-x)\rangle| + \sum_{k=1}^N \left|\left\langle c_t^k, \phi(.-x)\right\rangle\right|^2 \,\mathrm{d}t < \infty.$$

Let $(W^k_t, k = 1, \dots, N)_{t\geq 0}$ be independent scalar Brownian motions. We consider an equation of the form

$$\mathrm{d}u_t = b_t \,\mathrm{d}t + \sum_{k=1}^{N} c^k_t \,\mathrm{d}W^k_t. \tag{A.2}$$

Definition A.1. *A $\mathbb{D}$-valued stochastic process $(u_t)_{t\geq 0}$ is called a solution of* (A.2) in the sense of distributions *with initial condition u_0 if for $t \in (0, \infty)$ and $\phi \in C^\infty_0$*

$$\langle u_t, \phi\rangle - \langle u_0, \phi\rangle = \int_0^t \langle b_s, \phi\rangle \,\mathrm{d}t + \sum_{k=1}^{N} \int_0^t \left\langle c^k_s, \phi\right\rangle \mathrm{d}W^k_s \tag{A.3}$$

holds almost surely.

The following change of variable formula is a special case of a result by Krylov (2011, Theorem 1.1).

Theorem A.2 (generalized Itô–Wentzell formula). *Let $(u_t)_{t\geq 0}$ be a solution of* (A.2) *in the sense of distributions and let $(x_t)_{t\geq 0}$ be a locally integrable process with representation*

$$\mathrm{d}x_t = \mu_t \,\mathrm{d}t + \sum_{k=1}^{N} \sigma^k_t \,\mathrm{d}W^k_t, \qquad t \geq 0,$$

where $(\mu_t)_{t\geq 0}$ and $(\sigma^k_t, k = 1, \dots, N)_{t\geq 0}$ are real-valued predictable processes. Define the $\mathbb{D}$-valued process $(v_t)_{t\geq 0}$ by $v_t(x) := u_t(x + x_t)$ for $x \in \mathbb{R}$, $t \in [0, \infty)$. Then $(v_t)_{t\geq 0}$ is a solution of

$$\begin{aligned}\mathrm{d}v_t = &\left[b_t(. + x_t) + \tfrac{1}{2}\left(\sum_{k=1}^{N} \left|\sigma^k_t\right|^2\right) \tfrac{\partial^2}{\partial x^2} v_t + \mu_t \tfrac{\partial}{\partial x} v_t + \sum_{k=1}^{N} \left(\sigma^k_t \tfrac{\partial}{\partial x} c^k_t(. + x_t)\right)\right] \mathrm{d}t\\ &+ \sum_{k=1}^{N} \left[c^k_t(. + x_t) + \sigma^k_t \tfrac{\partial}{\partial x} v_t\right] \mathrm{d}W^k_t\end{aligned}$$

in the sense of distributions.

Remark A.3. It is worth noting that the correlation of (u_t) and (x_t) contributes the term

$$\sum_{k=1}^{N} \left(\sigma^k_t \tfrac{\partial}{\partial x} c^k_t(. + x_t)\right).$$

We now apply the above Itô–Wentzell formula in order to derive the dynamics of the order book density v, in noncentered coordinates, in the setting considered in sections 3 and 4.

Let $L \in (0, \infty]$ and $I := (-L, 0) \cup (0, L)$. For h, $f \in H^2(I) \cap H^1_0(I)$. Then, (1.2) with initial condition $u_0 = h$ admits a unique (analytically) strong solution denoted by $(u_t)_{t\geq 0}$. Let $\tilde{u}_t$ be the trivial extension of u_t to $\mathbb{R}$, i.e.,

$$\tilde{u}_t(x) := \begin{cases} u_t(x), & x \in I,\\ 0 & \text{otherwise.}\end{cases} \tag{A.4}$$

Note that $\tilde{u} \in H^2(\mathbb{R} \setminus \{-L, 0, L\}) \cap H^1(\mathbb{R})$. Recall that Δ and ∇ in the previous discussions denoted the weak derivatives on $\mathbb{R} \setminus \{-L, 0, L\}$, and we get that $\frac{\partial}{\partial x}\tilde{u} = \nabla\tilde{u}$ and

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\tilde{u} - \Delta\tilde{u} &= \frac{\partial}{\partial x}\nabla\tilde{u} - \nabla\nabla\tilde{u} \\ &= (\nabla\tilde{u}(-L+) - \nabla\tilde{u}(-L-))\delta_{-L} \\ &\quad + (\nabla\tilde{u}(0+) - \nabla\tilde{u}(0-))\delta_0 + (\nabla\tilde{u}(L+) - \nabla\tilde{u}(L-))\delta_L, \end{aligned} \tag{A.5}$$

where δ_x denotes a point mass at $x \in \mathbb{R}$. Define

$$b_t(x) := \begin{cases} \eta_a \Delta u_t(x) + \beta_a \nabla u_t(x) + \alpha_a u_t(x) + f_a(x), & x \in (0, L), \\ \eta_b \Delta u_t(x) - \beta_b \nabla u_t(x) + \alpha_b u_t(x) - f_b(x), & x \in (-L, 0), \\ 0 & \text{otherwise}, \end{cases} \tag{A.6}$$

$$c_t^1(x) := \begin{cases} \sigma_a \varrho_{a,b} u_t(x), & x \in (0, L), \\ \sigma_b u_t(x), & x \in (-L, 0), \\ 0 & \text{otherwise}, \end{cases} \tag{A.7}$$

$$c_t^2(x) := \begin{cases} \sigma_a \sqrt{1 - \varrho_{a,b}^2}\, u_t(x), & x \in (0, L), \\ 0 & \text{otherwise}, \end{cases} \tag{A.8}$$

so that

$$\mathrm{d}\tilde{u}_t = b_t \,\mathrm{d}t + c_t^1 \,\mathrm{d}W_t^1 + c_t^2 \,\mathrm{d}W_t^2. \tag{A.9}$$

The Cauchy–Schwarz inequality shows that (A.1) is satisfied. Assume now that the mid-price $(S_t)_{t \geq 0}$ follows the dynamics

$$\mathrm{d}S_t = c_s\theta\mu_t \,\mathrm{d}t + c_s\theta(\sigma_b - \sigma_a\varrho_{a,b}) \,\mathrm{d}W_t^1 - c_s\theta\sigma_a\sqrt{1 - \varrho_{a,b}^2} \,\mathrm{d}W_t^2 \tag{A.10}$$

for some integrable predictable process μ. Define

$$\sigma_s := c_s\theta\sqrt{\sigma_b^2 + \sigma_a^2 - 2\varrho_{a,b}\sigma_b\sigma_a}. \tag{A.11}$$

Then, Theorem A.2 yields that for $v_t(x) := \tilde{u}_t(x - S_t)$ we get

$$\begin{aligned} \mathrm{d}v_t = \Big[& b_t(. - S_t) + \tfrac{1}{2}\sigma_s^2 \tfrac{\partial^2}{\partial x^2} v_t - c_s\theta\mu_t \tfrac{\partial}{\partial x} v_t \\ & - \Big(c_s\theta\left(\sigma_b - \varrho_{a,b}\sigma_a\right) \tfrac{\partial}{\partial x} c_t^1(. - S_t) + c_s\theta\sqrt{1 - \varrho_{a,b}^2}\sigma_a \tfrac{\partial}{\partial x} c_t^2(. - S_t) \Big) \Big] \,\mathrm{d}t \\ & + \left(c_t^1(. - S_t) - c_s\theta\left(\sigma_b - \varrho_{a,b}\sigma_a\right) \tfrac{\partial}{\partial x} v_t \right) \mathrm{d}W_t^1 \\ & + \left(c_t^2(. - S_t) + c_s\theta\sqrt{1 - \varrho_{a,b}^2}\sigma_a \tfrac{\partial}{\partial x} v_t \right) \mathrm{d}W_t^2, \end{aligned} \tag{A.12}$$

i.e., v is a solution of the stochastic moving boundary problem,

$$
\begin{aligned}
\mathrm{d}v_t &= \Big[\left(\eta_a + \tfrac{1}{2}\sigma_s^2\right)\Delta v_t + \left(\beta_a - c_s\theta\mu_t - c_s\theta\left(\varrho_{a,b}\sigma_b\sigma_a - \sigma_a^2\right)\right)\nabla v_t \\
&\qquad\qquad\qquad\qquad + \alpha_a v_t + f_a(. - S_t)\Big]\,\mathrm{d}t \\
&\quad + \left(\sigma_a\varrho_{a,b}v_t - c_s\theta\left(\sigma_b - \varrho_{a,b}\sigma_a\right)\nabla v_t\right)\mathrm{d}W_t^1 \\
&\quad + \sigma_a\sqrt{1-\varrho_{a,b}^2}\left(v_t + c_s\theta\nabla v_t\right)\mathrm{d}W_t^2, \quad \text{on } (S_t, S_t+L), \\
\mathrm{d}v_t &= \Big[\left(\eta_b + \tfrac{1}{2}\sigma_s^2\right)\Delta v_t - \left(\beta_b + c_s\theta\mu_t + c_s\theta\left(\sigma_b^2 - \varrho_{a,b}\sigma_b\sigma_a\right)\right)\nabla v_t \\
&\qquad\qquad\qquad\qquad + \alpha_b v_t - f_b(. - S_t)\Big]\,\mathrm{d}t \\
&\quad + \left(\sigma_b v_t - c_s\theta\left(\sigma_b - \varrho_{a,b}\sigma_a\right)\nabla v_t\right)\mathrm{d}W_t^1 \\
&\quad + c_s\theta\sqrt{1-\varrho_{a,b}^2}\,\sigma_a\nabla v_t\,\mathrm{d}W_t^2 \quad \text{on } (S_t - L, S_t), \\
v_t &= 0 \qquad \text{otherwise}; \\
\mathrm{d}S_t &= c_s\theta\mu_t\,\mathrm{d}t + c_s\theta(\sigma_b - \varrho_{a,b}\sigma_a)\,\mathrm{d}W_t^1 - c_s\theta\sqrt{1-\varrho_{a,b}^2}\,\sigma_a\,\mathrm{d}W_t^2.
\end{aligned}
\tag{A.13}
$$

To define what we mean by solution in this context we introduce the mappings

$$
\begin{aligned}
\mathbb{L}\colon \bigcup_{x\in\mathbb{R}} &\left[\left(H^2(\mathbb{R}\setminus\{x-L,x,x+L\}) \cap H_0^1(\mathbb{R}\setminus\{x-L,x,x+L\})\right)\times\{x\}\right] \to \mathbb{D}, \\
&(v,s) \mapsto (\nabla(v(s-L+) - \nabla v(s-L-)))\delta_{s-L} \\
&\qquad + (\nabla(v(s+) - \nabla v(s-)))\delta_s \\
&\qquad + (\nabla(v(s+L+) - \nabla v(s+L-)))\delta_{s+L}.
\end{aligned}
$$

Define now the functions $\bar{\mu}\colon \mathbb{R}^5 \to \mathbb{R}$, $\bar{\sigma}_1$, $\bar{\sigma}_2\colon \mathbb{R}^4 \to \mathbb{R}$ as

$$
\bar{\mu}(x,y'',y',y,s) := \begin{cases} (\eta_a + \frac{1}{2}\sigma_s^2)y'' + (\beta_a - c_s\theta(\varrho_{a,b}\sigma_b\sigma_a - \sigma_a^2))y' \\ \qquad\qquad + \alpha_a y + f_a(x), & x \in (0,L), \\ (\eta_b + \frac{1}{2}\sigma_s^2)y'' - (\beta_a + c_s\theta(\sigma_b^2 - \varrho_{a,b}\sigma_b\sigma_a))y' \\ \qquad\qquad + \alpha_b y - f_b(x), & x \in (-L,0), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\bar{\sigma}_1(x,y',y,s) := \begin{cases} \sigma_a\varrho_{a,b}y, & x \in (0,L), \\ \sigma_b y, & x \in (-L,0), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\bar{\sigma}_2(x,y',y,s) := \begin{cases} \sigma_a\sqrt{1-\varrho_{a,b}^2}\,y, & x \in (0,L), \\ 0 & \text{otherwise} \end{cases}
$$

for x, y'', y', y, $s \in \mathbb{R}$.

Following Mueller (2018, Definition 1.11), a solution of (A.13) is an $L^2(\mathbb{R}) \times \mathbb{R}$-continuous stochastic process (v_t, S_t), taking values in

$$
\bigcup_{x\in\mathbb{R}} \left[\left(H^2(\mathbb{R}\setminus\{x-L,x,x+L\}) \cap H_0^1(\mathbb{R}\setminus\{x-L,x,x+L\})\right)\times\{x\}\right],
$$

such that (S_t) is given by (A.10) and, in the sense of distributions,

$$\mathrm{d}v_t = (\bar{\mu}(. - S_t, \Delta v_t, \nabla v_t, v_t, S_t))\ \mathrm{d}t - \nabla v_t\ \mathrm{d}S_t + \tfrac{1}{2}\mathbb{L}(v_t, S_t)\ \mathrm{d}\langle S\rangle_t + \bar{\sigma}_1(. - S_t, \nabla v_t, v_t, S_t)\ \mathrm{d}W_t^1 + \bar{\sigma}_2(. - S_t, \nabla v_t, v_t, S_t)\ \mathrm{d}W_t^2. \tag{A.14}$$

REFERENCES

B. M. Bibby and M. Sørensen (1995), *Martingale estimation functions for discretely observed diffusion processes*, Bernoulli, 1, pp. 17–39.

B. M. Bibby, I. M. Skovgaard, and M. Sørensen (2005), *Diffusion-type models with given marginal distribution and autocorrelation function*, Bernoulli, 11, pp. 191–220.

A. N. Borodin and P. Salminen (2012), *Handbook of Brownian Motion-Facts and Formulae*, Birkhäuser, Basel.

J.-P. Bouchaud, J. D. Farmer, and F. Lillo (2009), *How markets slowly digest changes in supply and demand*, in Handbook of Financial Markets: Dynamics and Evolution, T. Hens and K. R. Schenk-Hoppe, eds., Elsevier, Amsterdam, pp. 57–160.

M. Burger, L. Caffarelli, P. A. Markowich, and M.-T. Wolfram (2013), *On a Boltzmann-type price formation model*, Proc. A, 469, doi:10.1098/rspa.2013.0126.

L. A. Caffarelli, P. A. Markowich, and J.-F. Pietschmann (2011), *On a price formation free boundary model by Lasry and Lions*, C. R. Math., 349, pp. 621–624, doi:10.1016/j.crma.2011.05.011.

R. Carmona and K. Webster (2019), *The self-financing equation in limit order book markets*, Finance Stoch., 23, pp. 729–759.

A. Cartea, R. Donnelly, and S. Jaimungal (2018), *Enhancing trading strategies with order book signals*, Appl. Math. Finance, 25, pp. 1–35.

J. A. Chavez-Casillas and J. Figueroa-Lopez (2017), *A one-level limit order book model with memory and variable spread*, Stochastic Process. Appl., 127, pp. 2447–2481, doi:10.1016/j.spa.2016.11.005.

L. Chayes, M. del Mar Gonzalez, M. Gualdani, and I. Kim (2009), *Global existence and uniqueness of solutions to a model of price formation*, SIAM J. Math. Anal., 41, pp. 2107–2135, doi:10.1137/090753346.

R. Cont, A. Kukanov, and S. Stoikov (2014), *The price impact of order book events*, J. Financial Econometrics, 12, pp. 47–88, doi:10.1093/jjfinec/nbt003.

R. Cont (2005), *Modeling term structure dynamics: An infinite dimensional approach*, Int. J. Theor. Appl. Finance, 8, pp. 357–380.

R. Cont (2011), *Statistical modeling of high-frequency financial data*, IEEE Signal Process., 28, pp. 16–25.

R. Cont and A. de Larrard (2012), *Order Book Dynamics in Liquid Markets: Limit Theorems and Diffusion Approximations*, http://ssrn.com/abstract=1757861.

R. Cont and A. de Larrard (2013), *Price dynamics in a Markovian limit order market*, SIAM J. Financial Math., 4, pp. 1–25, doi:10.1137/110856605.

R. Cont and M. Mueller (2018), *LOBPY Python Package*, https://github.com/mm842/lobpy.

R. Cont, S. Stoikov, and R. Talreja (2010), *A stochastic model for order book dynamics*, Oper. Res., 58, pp. 549–563, doi:10.1287/opre.1090.0780.

G. Da Prato and J. Zabczyk (2014), *Stochastic Equations in Infinite Dimensions*, 2nd ed., Cambridge University Press, Cambridge, UK, doi:10.1017/CBO9781107295513.

L. C. Evans *Partial Differential Equations* (2010), 2nd ed., Grad. Stud. Math. 19, AMS, Providence, RI.

D. Filipovic and J. Teichmann (2003), *Existence of invariant manifolds for stochastic equations in infinite dimension*, J. Funct. Anal., 197, pp. 398–432, https://doi.org/10.1016/S0022-1236(03)00008-9.

J. L. Forman and M. Sørensen (2008), *Pearson diffusions: A class of statistically tractable diffusion processes*, Scand. J. Stat., 35, pp. 438–465.

R. Gaspar (2006), *Finite dimensional markovian realizations for forward price term structure models*, in Stochastic Finance, A. N. Shiryaev, M. R. Grossinho, P. E. Oliveira, and M. L. Esquivel, eds., Springer, New York, pp. 265–320.

B. Hambly, J. Kalsi, and J. Newbury (2020), *Limit order books, diffusion approximations and reflected SPDEs: From microscopic to macroscopic models*, Appl. Math. Finance, 27, pp. 132–170, doi:10.1080/1350486X.2020.1758176.

U. Horst and D. Kreher (2018), *Second order approximations for limit order books*, Finance Stoch., 22, pp. 827–877, doi:10.1007/s00780-018-0373-7.

W. Huang, C.-A. Lehalle, and M. Rosenbaum (2017), *Simulating and analyzing order book data: The queue-reactive model*, J. Amer. Statist. Assoc., 110, pp. 107–122.

O. Kallenberg (2002), *Foundations of Modern Probability*, Springer, New York.

I. Karatzas and C. Kardaras (2007), *The numéraire portfolio in semimartingale financial models*, Finance Stoch., 11, pp. 447–493.

I. Karatzas and S. Shreve (1987), *Brownian Motion and Stochastic Calculus*, Springer, New York.

F. Kelly and E. Yudovina (2018), *A Markov model of a limit order book: Thresholds, recurrence, and trading strategies*, Math. Oper. Res., 43, pp. 181–203, doi:10.1287/moor.2017.0857.

A. Kirilenko, A. Kyle, M.Samadi, and T. Tuzun (2017), *The flash crash: High frequency trading on an electronic market*, J. Finance, 72, pp. 967–998.

P. E. Kloeden and E. Platen (1992), *Numerical Solution of Stochastic Differential Equations*, Springer, New York.

N. V. Krylov (2011), *On the Itô-Wentzell formula for distribution-valued processes and related topics*, Probab. Theory Related Fields, 150, pp. 295–319, doi:10.1007/s00440-010-0275-x.

J.-M. Lasry and P.-L. Lions (2007), *Mean field games*, Jpn. J. Math., 2, pp. 229–260, doi:10.1007/s11537-007-0657-8.

C.-A. Lehalle and S. Laruelle (2018), *Market Microstructure in Practice*, World Scientific, River Edge, NJ.

N. N. Leonenko and N. Šuvak (2010), *Statistical inference for reciprocal gamma diffusion process*, J. Statistical Plann. Inference, 140, pp. 30–51.

J. Lévine (1991), *Finite dimensional realizations of stochastic PDEs and application to filtering*, Stoch. Stoch. Rep., 37, pp. 75–103, doi:10.1080/17442509108833728.

A. Lipton, U. Pesavento, and M. G Sotiropoulos (2014), *Trading strategies via book imbalance*, RISK, pp. 70–75.

H. Luckock (2003), *A steady-state model of the continuous double auction*, Quant. Finance, 3, pp. 385–404, doi:10.1088/1469-7688/3/5/305.

P. Markowich, J. Teichmann, and M. Wolfram (2016), *Parabolic free boundary price formation models under market size fluctuations*, Multiscale Model. Simul., 14, pp. 1211–1237, doi:10.1137/16M1066099.

A. Milian (2002), *Comparison theorems for stochastic evolution equations*, Stoch. Stoch. Rep., 72, pp. 79–108.

M. S. Mueller (2018), *A stochastic Stefan-type problem under first-order boundary conditions*, Ann. Appl. Probab., 28, pp. 2335–2369.

D. Revuz and M. Yor (1999), *Continuous Martingales and Brownian Motion*, Springer, New York.

H. Schlichting and K. Gersten (2017), *Boundary-layer Theory*, Grad. Texts in Math., Springer, New York.

E. Smith, J. D. Farmer, L. Gillemot, and S. Krishnamurthy (2003), *Statistical theory of the continuous double auction*, Quant. Finance, 3, pp. 481–514.

K. Yosida (1995), *Functional Analysis*, Classics in Math., Springer, New York.