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A functional extension of the Ito formula

Une extension fonctionnelle de la formule d'Ito

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ABSTRACT

We develop a non-anticipative pathwise calculus for functionals of a Brownian semimartingale and its quadratic variation. A functional Ito formula is obtained for locally Lipschitz functionals of a Brownian semimartingale and its quadratic variation. As a result we obtain a constructive martingale representation theorem for Brownian martingales verifying a regularity property.

RÉSUMÉ

Nous esquissons un calcul fonctionnel non anticipatif pour des fonctionelles d'une semi-martingale Brownienne et sa variation quadratique. Nous montrons, pour des fonctionnelles vérifiant une propriété de Lipschitz locale, une formule de changement de variable qui généralise la formule d'Ito. Ce résultat permet d'obtenir une version constructive du théorème de représentation de martingale pour une classe de fonctionnelles Browniennes.

Version française abrégée

Considérons un processus d'Ito $X(t)=\int_0^t \mu(u)\,\mathrm{d}u+\int_0^t \sigma(u)\,\mathrm{d}W_u$ où W est un mouvement Brownien d-dimensionnel défini sur un espace de probabilité filtré $(\Omega_0,\mathcal{B},\mathcal{B}_t,\mathbb{P})$ et μ,σ des processus càdlàg adaptés à $\mathcal{B}_t$. Nous noterons $X(t)$ la valeur du processus à l'instant t et $X_t=(X(u),0\leqslant u\leqslant t)$ sa trajectoire sur $[0,t]$. Notons $(\mathcal{F}_t)_{t\geqslant 0}$ la filtration naturelle de X et $[X](t)=\int_0^t A(u)\,\mathrm{d}u=\int_0^t {}^t\sigma(u).\sigma(u)\,\mathrm{d}u$ sa variation quadratique, à valeurs dans l'ensemble S_d^+ des matrices symétriques positives. Un processus réel $Y=(Y(t))_{t\in[0,T_*[}$ adapté à $\mathcal{F}_t$ peut être vu comme une famille de fonctionnelles $Y(t,.)$ sur l'espace $D([0,T_*[,\mathbb{R}^d)$ des fonctions càdlàg. Nous considérons des processus Y qui peuvent se représenter comme

$$Y(t)=F_t\big(\{X(u),0\leqslant u\leqslant t\},\{A(u),0\leqslant u\leqslant t\}\big)=F_t(X_t,A_t)$$

où $(F_t)_{t\geqslant 0}$ est une familllle de fonctionnelles $F_t: D([0,t],\mathbb{R}^d)\times D([0,t],S_d^+)\to\mathbb{R}$ représentant la dépendance de $Y(t)$ par rapport à la trajectoire de X et de sa variation quadratique. F peut être vu comme une fonctionnelle défini sur le fibré vectoriel $\Upsilon=\bigcup_{t\in[0,T_*[}D([0,t],\mathbb{R}^d)\times D([0,t],S_d^+)$. Nous définissons une distance sur Υ, induite par la norme sup sur les trajectoires, ainsi que la classe $\mathbb{F}^\infty$ des fonctionnelles localement Lipschitz par rapport à cette distance (Définition 2.1). Si $F\in\mathbb{F}^\infty$ alors $Y(t)=F(X_t,A_t)$ définit un processus optionnel aux trajectoires càdlàg (Théorème 2.3).

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Notons pour $x \in ([0,T],\mathbb{R}^d)$, x_t la restriction de x à $[0,t]$, $x_{t,h}$ le prolongement continu par une constante de x_t à $[0,t+h]$ et $x_t^h = x_t + h\ 1_t$ la fonction càdlàg obtenu en ajoutant un saut de taille h en t. La dérivée en temps (ou dérivée « horizontale ») d'une fonctionnelle $F = (F_t)_{t\in[0,T_*[}$ sur Υ est définie par (Définition 3.1)

$$\mathcal{D}_t F\ (x,v) = \lim_{h\to 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x_t, v_t)}{h}, \quad x_t \in D\big([0,t],\mathbb{R}^d\big), v_t \in D\big([0,t], S_d^+\big)$$

et la notion de dérivée « verticale » (Définition 3.2)

$$\nabla_x F_t(x_t,v_t) = \big(\partial_i F_t(x_t,v_t),\ i=1,\dots,d\big) \quad \text{avec } \partial_i F_t(x_t,v_t) = \lim_{h\to 0} \frac{F_t(x_t^{he_i}, v_t) - F_t(x_t,v_t)}{h}$$

Si $Y(t) = F_t(X_t, A_t)$ où $\nabla_x F, \mathcal{D}_t F \in \mathbb{F}^\infty$ nous montrons (Théorème 4.1) que les processus

$$\nabla_X Y(t) = \nabla_x F(X_t, A_t), \qquad \mathcal{D}Y(t) = \mathcal{D}_t F(X_t, A_t),$$

appelées resp. dérivée horizontale de Y et dérivée (verticale) de Y le long de X, ne dépendent pas du choix de F. Nous montrons alors (Théorème 4.2) une formule de changement de variable fonctionnelle, analogue à la formule d'Ito : si $Y(t) = F_t(X_t, A_t)$ et si $F \in \mathbb{F}^{\infty,1}$ et ses dérivées horizontales et verticales d'ordre un et deux sont dans $\mathbb{F}^\infty$ alors

$$Y(T) - Y(0) = \int_0^T \mathcal{D}Y(t)\,\mathrm{d}t + \int_0^T \nabla_X Y(t)\,\mathrm{d}X(t) + \frac{1}{2}\int_0^T \mathrm{tr}\big[{}^t\nabla_X^2 Y(t) A(t)\big]\,\mathrm{d}t \quad \text{p.s.}$$

Ce résultat permet d'obtenir une version constructive du théorème de représentation de martingale (Théorème 5.1) : si $X(t) = \int_0^t \sigma\,\mathrm{d}W$ est une martingale Brownienne alors toute $\mathcal{F}_t$-martingale Y vérifiant les hypothèse ci-dessus admet une représentation intégrale donnée par

$$Y_T = E[Y_T] + \int_0^T \nabla_X Y(t)\,\mathrm{d}X(t) = E[Y_T] + \int_0^T \nabla_X Y(t)\sigma(t)\,\mathrm{d}W(t)$$

Ce résultat permet en particulier d'obtenir une version non anticipative de la formule de Clark–Ocone, dans laquelle la projection prévisible de la dérivée de Malliavin est remplacée par une dérivée verticale.

1. Introduction

Let $X : [0,T_*[\times \Omega \mapsto \mathbb{R}^d$ be a d-dimensional Ito process, i.e. a continuous, $\mathbb{R}^d$-valued semimartingale defined on a filtered probability space $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$ which admits a stochastic integral representation

$$X(t) = \int_0^t \mu(u)\,\mathrm{d}u + \int_0^t \sigma(u)\,\mathrm{d}W_u \tag{1}$$

where W is a d-dimensional Brownian motion and $\mu(t)$, $\sigma(t)$ are cadlag adapted processes with

$$E\int_0^T \|\mu(u)\|\,\mathrm{d}u < \infty, \qquad E\int_0^T \|\sigma(u)\|^2\,\mathrm{d}u < \infty$$

for $T < T_*$. Denote by $[X](t) = \int_0^t A(u)\,\mathrm{d}u$ the quadratic variation process of X, where $A = {}^t\sigma.\sigma$ takes values in the cone S_d^+ of symmetric positive $d\times d$ matrices and $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \vee \mathcal{N})$ the natural filtration of X augmented by the null sets.

The paths of X then lie in $(C_0([0,T_*[,\mathbb{R}^d)$, the space of continuous functions, which we will view as a subspace of the Skorokhod space $D([0,T_*[,\mathbb{R}^d)$ of cadlag functions. For a path $x \in D([0,T],\mathbb{R}^d)$, denote by $x(t)$ the value of x at t and by $x_t = (x(u), 0 \leqslant u \leqslant t)$ the restriction of x to $[0,t]$. Thus $x_t \in D([0,t],\mathbb{R}^d)$. For a process X we shall similarly denote $X(t)$ its value and $X_t = (X(u), 0 \leqslant u \leqslant t)$ its path on $[0,t]$.

A process $Y = (Y(t))_{t\in[0,T_*[}$ which is progressively measurable with respect to $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \vee \mathcal{N})$ may be represented as a (jointly) measurable family of functionals $Y(t, X_u(\omega), u \in [0,t])$ where $Y(t,.) : D([0,t],\mathbb{R}^d) \mapsto \mathbb{R}$ represents the dependence of $Y(t)$ on the underlying path of X on $[0,t]$. This functional approach to stochastic processes [1,4,8,7,10] has proven fruitful for investigating various properties of Brownian functionals. The framework of Malliavin calculus [8,7,10] gives a powerful tool for analyzing smooth functionals in the Fréchet or Malliavin sense and yields explicit formulas for stochastic integral representation of such functionals in terms of Malliavin derivatives [6,9,8]. While the notion of Malliavin derivative leads to integral representations in terms of *anticipative* quantities [1,5,9,8], in many applications it is more natural to consider non-anticipative/causal versions of such representations. Also, many examples of functionals arising in statistics of processes, physics or mathematical finance of the form

$$\int_0^t g(t, X_t) d[X](t), \quad G\big(t, X_t, [X]_t\big) \quad \text{with } G, g \text{ smooth} \tag{2}$$

are not smooth with respect to the underlying path of X in the supremum norm or L^p-norms.

B. Dupire [3] has recently defined a notion of pathwise derivative for adapted functionals of a semimartingale. Following the ideas in [3], we develop a non-anticipative calculus for a class of functionals – including the above examples – which may be represented as

$$Y(t) = F_t\big(\{X(u), 0 \leqslant u \leqslant t\}, \{A(u), 0 \leqslant u \leqslant t\}\big) = F_t(X_t, A_t) \tag{3}$$

where $A = {}^t\sigma.\sigma$ and the family of functionals $F_t : D([0,t], \mathbb{R}^d) \times D([0,t], S_d^+) \to \mathbb{R}$ represents the dependence of Y on the underlying path. For such functionals, we define an appropriate notion of (Lipschitz) regularity and a non-anticipative concept of pathwise derivative. Using these pathwise derivatives, we derive a functional Ito formula (Theorem 4.2) which extends the usual Ito formula in two ways: it allows for path-dependence and for dependence with respect to quadratic variation process. We use this result to derive a constructive version of the martingale representation theorem (Theorem 5.1), which can be seen as a non-anticipative form of the Clark–Haussmann–Ocone formula [1,5,9]. Our results give a rigorous justification to the ideas first explored by B. Dupire [3] and extend them to a larger class of functionals which notably allow for dependence on the quadratic variation along a path. This note summarizes the main results; complete proofs are given in [2].

2. Functional representation of non-anticipative processes

Consider an Ito process X as in (1) and denote $[X](t) = \int_0^t {}^t\sigma(u).\sigma(u)\,du = \int_0^t A(u)\,du$ its quadratic (co-)variation process. We shall assume that the process $A(t) = {}^t\sigma(t).\sigma(t)$, which takes values in the set S_d^+ of symmetric positive $d \times d$ matrices, has cadlag paths. Note that A need not be a semimartingale. Denote by $D([0,t], \mathbb{R}^d)$ (resp. $\mathcal{S}_t^+ = D([0,t], S_d^+)$) the space of cadlag functions with values in $\mathbb{R}^d$ (resp. S_d^+) and by $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \vee \mathcal{N})$ the natural filtration of X augmented by the null sets. An $\mathcal{F}_t$-progressively measurable process $Y : [0, T_*[\times \Omega \mapsto \mathbb{R}^d$ may be represented as

$$Y(t) = F_t\big(\{X(u), 0 \leqslant u \leqslant t\}, \{A(u), 0 \leqslant u \leqslant t\}\big) = F_t(X_t, A_t) \tag{4}$$

where $F = (F_t)_{t \in [0,T_*[}$ is a jointly measurable family of functionals

$$F_t : D\big([0,t], \mathbb{R}^d\big) \times \mathcal{S}_t^+ \to \mathbb{R}$$

representing the dependence of Y_t on the underlying path of X on $[0,t]$ and its quadratic variation. $F = (F_t)_{t \in [0,T_*[}$ may be viewed as a functional on the vector bundle $\Upsilon = \bigcup_{t \in [0,T_*[} D([0,t], \mathbb{R}^d) \times D([0,t], S_d^+)$. Introducing A_t as additional variable allows to control the dependence of Y with respect to the “quadratic variation” $[X]$ by requiring smoothness of F_t with respect to the second variable in L^p-norms.

Consider now a path $x \in D([0, T_*[, \mathbb{R}^d)$ and denote by $x_t \in D([0,t], \mathbb{R}^d)$ its restriction to $[0,t]$. For $h \geqslant 0$, the *horizontal* extension $x_{t,h} \in D([0,t+h], \mathbb{R}^d)$ of x_t to $[0,t+h]$ is defined as

$$x_{t,h}(u) = x(u), \quad u \in [0,t]; \qquad x_{t,h}(u) = x(t), \quad u \in]t, t+h] \tag{5}$$

For $h \in \mathbb{R}^d$, we define the *vertical* extension x_t^h of x_t as the cadlag path obtained by shifting the endpoint:

$$x_t^h(u) = x_t(u), \quad u \in [0,t[, \qquad x_t^h(t) = x(t) + h, \quad \text{i.e. } x_t^h(u) = x_t(u) + h 1_{t=u} \tag{6}$$

We now define two metrics which extend the supremum/L^1-norms to paths of different length. For $T_* \geqslant t' = t+h \geqslant t \geqslant 0$, $(x,v) \in D([0,t], \mathbb{R}^d) \times S_t^+$ and $(x',v') \in D([0,t+h], \mathbb{R}^d) \times \mathcal{S}_{t+h}^+$ define

$$d_\infty\big((x,v),(x',v')\big) = \sup_{u \in [0,t+h]} \big|x_{t,h}(u) - x'(u)\big| + \sup_{u \in [0,t+h]} \big|v_{t,h}(u) - v'(u)\big| + h \tag{7}$$

$$d_{\infty,1}\big((x,v),(x',v')\big) = \sup_{u \in [0,t+h]} \big|x_{t,h}(u) - x'(u)\big| + \int_0^{t+h} \big|v_{t,h}(u) - v'(u)\big|\,du + h \tag{8}$$

Definition 2.1. Define the set $\mathbb{F}^\infty$ of functionals $F = (F_t, t \in [0, T_*[)$ on Υ which are locally Lipschitz for the d_∞ metric, uniformly on $[0, T_*[$: for any compact $K \subset \mathbb{R}^d$ and $R > 0$, there exists $C > 0$ such that

$$\begin{aligned} &\forall x \in D\big([0,t], K\big), \quad \forall x' \in D\big([0,t+h], K\big), \quad \forall v \in \mathcal{S}_t^+, \forall v \in \mathcal{S}_{t+h}^+, \\ &\|v\|_\infty \leqslant R, \quad \|v'\|_\infty \leqslant R \quad \Rightarrow \quad \big|F_t(x,v) - F_{t+h}(x',v')\big| \leqslant C d_\infty\big((x,v),(x',v')\big) \end{aligned} \tag{9}$$

Define $\mathbb{F}^{\infty,1}$ as the set of functionals $F = (F_t, t \in [0, T_*[)$ on Υ which are locally Lipschitz for the $d_{\infty,1}$ metric, uniformly on $[0, T_*[$: for any compact set $K \subset \mathbb{R}^d$ and any $R > 0$, there exists $C > 0$ such that

$$\begin{aligned}&\forall x \in D\big([0,t],K\big), \quad \forall x' \in D\big([0,t+h],K\big), \quad \forall v \in \mathcal{S}_t^+, \quad \forall v \in \mathcal{S}_{t+h}^+,\\ &\|v\|_\infty \leqslant R, \|v'\|_\infty \leqslant R \quad \Rightarrow \quad \big|F_t(x,v) - F_{t+h}\big(x',v'\big)\big| \leqslant C d_{\infty,1}\big((x,v),\big(x',v'\big)\big)\end{aligned} \tag{10}$$

The following results describe the processes generated by this class of functionals:

Lemma 2.2 *(Preservation of continuity and the cadlag property). If $F \in \mathbb{F}^\infty$ then for any $(x,v) \in D([0,T],\mathbb{R}^d) \times D([0,T],S_d^+)$, the path $t \mapsto F_t(x_t, v_t)$ is cadlag. If furthermore $F \in \mathbb{F}^{\infty,1}$ then the path $t \mapsto F_t(x_t, v_t)$ is continuous at all points where x is continuous.*

Theorem 2.3. *If $F \in \mathbb{F}^\infty$ then $Y(t) = F_t(X_t, A_t)$ defines an optional process.*

3. Horizontal and vertical derivatives

Let $(x,v) \in D([0,T_*[,\mathbb{R}^d) \times D([0,T_*[,S_d^+)$ and $t < T_*$ and $F = (F_t)_{t\in[0,T_*[} \in \mathbb{F}^\infty$.

Definition 3.1 *(Horizontal derivative).* $F \in \mathbb{F}^\infty$ has horizontal derivative $\mathcal{D}_t F(x,v)$ at (x,v) if

$$\mathcal{D}_t F(x,v) = \lim_{h\to 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x_t, v_t)}{h}, \quad x_t \in D\big([0,t],\mathbb{R}^d\big), v_t \in D\big([0,t],S_d^+\big) \tag{11}$$

This is a "Lagrangian" derivative of the functional F computed along the path. It was introduced by B. Dupire (in a slightly different form) [3] for computing the time decay of a financial option.

Definition 3.2 *(Dupire derivative).* The vertical derivative of $F \in \mathbb{F}^\infty$ at $(x_t, v_t) \in D([0,t],\mathbb{R}^d) \times D([0,t],S_d^+)$ is defined as

$$\nabla_x F_t\,(x,v) = \big(\partial_i F_t(x,v),\ i = 1,\ldots,d\big) \quad \text{where } \partial_i F_t(x,v) = \lim_{h\to 0} \frac{F_t(x^{he_i}, v) - F_t(x,v)}{h} \tag{12}$$

where $(e_i, i = 1,\ldots,d)$ is the canonical basis in $\mathbb{R}^d$.

Remark 1. $\partial_i F_t(x,v)$ is simply the directional derivative of F_t in direction $(1_{[t,\infty[}e_i, 0)$. Note that this involves examining cadlag perturbations of the path x, even if x is continuous.

Remark 2. If $F_t(x,v) = f(t, x(t))$ with $f \in C^{1,1}([0,T_*[\times \mathbb{R}^d)$ then we retrieve the usual partial derivatives:

$$D_t F_t(x,v) = \partial_t f\big(t, X(t)\big), \quad \nabla_x F_t(X_t, A_t) = \nabla_x f\big(t, X(t)\big)$$

Note that, unlike the definition of a Fréchet (resp. Malliavin) derivative in which F is perturbed along all directions in $C_0([0,T],\mathbb{R}^d)$ (resp. in the Cameron–Martin space H^1), we only examine *local* perturbations, so $\nabla_x F$ and $\mathcal{D}_t F$ seem to contain *less* information on the behavior of the functional F. Nevertheless we will show now that these derivatives are sufficient to reconstitute the path of $Y(t) = F_t(X_t, A_t)$.

4. Functional Ito formula

Consider an $\mathcal{F}_t$-adapted process $(Y(t))_{t\in[0,T_*[}$ given by a functional representation $Y(t) = F_t(X_t, A_t)$ where $F \in \mathbb{F}^{\infty,1}$ has horizontal and vertical derivatives $\mathcal{D}_t F \in \mathbb{F}^\infty$ and $\nabla_x F \in \mathbb{F}^\infty$.

Theorem 4.1. *Assume $F^1, F^2 \in \mathbb{F}^\infty$ admit vertical and horizontal derivatives $\nabla_x F^1, \mathcal{D}_t F^1, \nabla_x F^2, \mathcal{D}_t F^2 \in \mathbb{F}^\infty$. If F^1 and F^2 coincide on continuous paths*:

$$\begin{aligned}&\forall t \leqslant T_*,\ \forall (x,v) \in C_0\big([0,t],\mathbb{R}^d\big) \times D\big([0,t],S_d^+\big), \qquad F_t^1(x,v) = F_t^2(x,v)\\ &\textit{then} \quad \forall t \leqslant T_*,\ \forall (x,v) \in C_0\big([0,t],\mathbb{R}^d\big) \times D\big([0,t],S_d^+\big), \qquad \nabla_x F_t^1(x,v) = \nabla_x F_t^2(x,v)\end{aligned}$$

This allows one to define (independently of the choice of F) the processes

$$\nabla_X Y(t) := \nabla_x F_t(X_t, A_t) \quad \text{and} \quad \mathcal{D}_t Y(t) := D_t F(X_t, A_t) \tag{13}$$

$\nabla_X Y$ (resp. $\mathcal{D}_t Y$) is a $\mathbb{R}^d$-valued (resp. real-valued) optional process with cadlag paths.

Theorem 4.2 *(Functional Ito formula). Let Y be given by $Y(t) = F_t(X_t, A_t)$ where*

$$F \in \mathbb{F}^{\infty,1}, \quad \mathcal{D}_t F \in \mathbb{F}^{\infty} \quad \nabla_x F \in \mathbb{F}^{\infty}, \quad \nabla_x^2 F \in \mathbb{F}^{\infty} \tag{14}$$

$$\text{then} \quad Y(T) - Y(0) = \int_0^T \mathcal{D}_t Y(t)\,\mathrm{d}t + \int_0^T \nabla_X Y(t).\,\mathrm{d}X(t) + \frac{1}{2}\int_0^T \operatorname{tr}\left[{}^t\nabla_X^2 Y(t)\; A(t)\right]\mathrm{d}t \quad \text{a.s.} \tag{15}$$

- If $F_t(X_t, A_t) = f(t, X(t))$ where $f \in C^{1,2}([0, T_*[\times\mathbb{R}^d)$ this reduces to the standard Ito formula.
- As expected, Y depends on F and its derivatives only via their values on continuous paths. More precisely, Y can be reconstructed from the second-order jet of F on $\mathcal{C} = \bigcup_{t\in[0,T_*[} C_0([0,t], \mathbb{R}^d) \times C_0([0, T_*[, S_d^+) \subset \Upsilon$.

5. Martingale representation formula

Consider now the case where $X(t) = \int_0^t \sigma(t)\,dW(t)$ is a Brownian martingale. Consider an $\mathcal{F}_T$-measurable functional $H = H(X(t), t \in [0,T]) = H(X_T)$ with $E[|H|^2] < \infty$ and define the martingale $Y(t) = E[H|\mathcal{F}_t]$. The martingale representation theorem then states that there exists a predictable process ϕ such that $H = E[H] + \int_0^T \phi(t)\,dX(t)$. Using Theorem 4.2, we obtain a representation for ϕ in terms of the pathwise derivative of the martingale Y:

Theorem 5.1. *If there exists $F = (F_t)_{t\in[0,T]} \in \mathbb{F}^{\infty}$ verifying* (14) *such that $Y(t) = F_t(X_t, A_t)$ then*

$$H = E[H] + \int_0^T \nabla_X Y(t)\,\mathrm{d}X(t) = E[H] + \int_0^T \nabla_X Y(t)\sigma(t)\,\mathrm{d}W(t) \tag{16}$$

Note that regularity assumptions are given not on $H = Y(T)$ but on the functionals $Y(t) = E[H|\mathcal{F}_t]$, which is usually much more regular than H itself. If furthermore H is differentiable in the Malliavin sense [7,8,10], e.g. $H \in \mathbf{D}^{2,1}$ with Malliavin derivative $\mathbb{D}_t H$, then the Clark–Haussmann–Ocone formula [6,9,8] gives a representation of the integrand ϕ as the predictable projection of the Malliavin derivative:

$$H = E[H] + \int_0^T {}^p E[\mathbb{D}_t H|\mathcal{F}_t]\,\mathrm{d}W_t \tag{17}$$

Comparing (16) with (17) implies that $(\nabla_X Y)(t)\sigma(t)$ is a version of $E[\mathbb{D}_t H|\mathcal{F}_t]$. In the case where $X = W$ is a Brownian motion we obtain the following identity linking the two derivatives

$$\nabla_W E[H|\mathcal{F}_t] = E[\mathbb{D}_t H|\mathcal{F}_t] \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-a.e.} \tag{18}$$

Unlike $E[\mathbb{D}_t H|\mathcal{F}_t]$, $\nabla_X Y$ only involves non-anticipative quantities which may be computed pathwise.

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