Chapter 8

Functional Kolmogorov equations

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One of the key topics in Stochastic Analysis is the deep link between Markov processes and partial differential equations, which can be used to characterize a diffusion process in terms of its infinitesimal generator [69]. Consider a second-order differential operator $L\colon C_b^{1,2}([0,T]\times\mathbb{R}^d)\to C_b^0([0,T]\times\mathbb{R}^d)$ defined, for test functions $f\in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, by

$$Lf(t,x)=\frac{1}{2}\mathrm{tr}\big(\sigma(t,x)\cdot^{\mathrm{t}}\sigma(t,x)\partial_x^2 f\big)+b(t,x)\partial_x f(t,x),$$

with continuous, bounded coefficients

$$b\in C_b^0\big([0,T]\times\mathbb{R}^d,\mathbb{R}^d\big),\quad \sigma\in C_b^0\big([0,T]\times\mathbb{R}^d,\mathbb{R}^{d\times n}\big).$$

Under various sets of assumptions on b and σ (such as Lipschitz continuity and linear growth; or uniform ellipticity, say, $\sigma^{\mathrm{t}}\sigma(t,x)\geq\epsilon I_d$), the stochastic differential equation

$$dX(t)=b(t,X(t))dt+\sigma(t,X(t))dW(t)$$

has a unique weak solution $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},W,\mathbb{P})$ and X is a Markov process under $\mathbb{P}$ whose evolution operator $(P^X_{t,s},s\geq t\geq0)$ has infinitesimal generator L. This weak solution $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},W,\mathbb{P})$ is characterized by the property that, for every $f\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$,

$$f(u,X(u))-\int_0^u(\partial_t f+Lf)(t,X(t))dt$$

is a $\mathbb{P}$-martingale [69]. Indeed, applying Itô's formula yields

$$\begin{aligned}f(u,X(u))=f(0,X(0))&+\int_0^u(\partial_t f+Lf)(t,X(t))dt\\&+\int_0^u\partial_x f(t,X(t))\sigma(t,X(t))dW.\end{aligned}$$

In particular, if $(\partial_t f+Lf)(t,x)=0$ for every $t\in[0,T]$ and every $x\in\mathrm{supp}(X(t))$, then $M(t)=f(t,X(t))$ is a martingale.

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More generally, for $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$, $M(t) = f(t, X(t))$ is a local martingale if and only if f is a solution of

$$\partial_t f(t,x) + Lf(t,x) = 0 \quad \forall t \geq 0, \quad x \in \operatorname{supp}(X(t)).$$

This PDE is the Kolmogorov (backward) equation associated with X, whose solutions are the *space-time* (L-)*harmonic functions* associated with the differential operator L: f is space-time harmonic if and only if $f(t, X(t))$ is a (local) martingale.

The tools from Functional Itô Calculus may be used to extend these relations beyond the Markovian setting, to a large class of processes with path dependent characteristics. This leads us to a new class of partial differential equations on path space —*functional Kolmogorov equations*— which have many properties in common with their finite-dimensional counterparts and lead to new Feynman–Kac formulas for path dependent functionals of semimartingales. This is an exciting new topic, with many connections to other strands of stochastic analysis, and which is just starting to be explored [10, 14, 23, 28, 60].

8.1 Functional Kolmogorov equations and harmonic functionals

8.1.1 Stochastic differential equations with path dependent coefficients

Let $(\Omega = D([0,T], \mathbb{R}^d), \mathcal{F}^0)$ be the canonical space and consider a semimartingale X which can be represented as the solution of a stochastic differential equation whose coefficients are allowed to be path dependent, left continuous functionals:

$$dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t), \tag{8.1}$$

where b and σ are non-anticipative functionals with values in $\mathbb{R}^d$ (resp., $\mathbb{R}^{d\times n}$) whose coordinates are in $\mathbb{C}_l^{0,0}(\Lambda_T)$ such that equation (8.1) has a unique weak solution $\mathbb{P}$ on $(\Omega, \mathcal{F}^0)$.

This class of processes is a natural 'path dependent' extension of the class of diffusion processes; various conditions (such as the functional Lipschitz property and boundedness) may be given for existence and uniqueness of solutions (see, e.g., [62]). We give here a sample of such a result, which will be useful in the examples.

Proposition 8.1.1 (Strong solutions for path dependent SDEs)**.** *Assume that the non-anticipative functionals b and σ satisfy the following Lipschitz and linear growth conditions:*

$$\left|b(t,\omega) - b(t,\omega')\right| + \left|\sigma(t,\omega) - \sigma(t,\omega')\right| \leq K \sup_{s\leq t} \left|\omega(s) - \omega'(s)\right| \tag{8.2}$$

and

$$\big|b(t,\omega)\big| + \big|\sigma(t,\omega)\big| \le K\Big(1 + \sup_{s\le t}|\omega(s)| + |t|\Big), \tag{8.3}$$

for all $t \ge t_0$, $\omega, \omega' \in C^0([0,t],\mathbb{R}^d)$. *Then, for any* $\xi \in C^0([0,T],\mathbb{R}^d)$, *the stochastic differential equation* (8.1) *has a unique strong solution* X *with initial condition* $X_{t_0} = \xi_{t_0}$. *The paths of* X *lie in* $C^0([0,T],\mathbb{R}^d)$ *and*

(i) *there exists a constant* C *depending only on* T *and* K *such that, for* $t \in [t_0,T]$,

$$E\Big[\sup_{s\in[0,t]}|X(s)|^2\Big] \le C\Big(1 + \sup_{s\in[0,t_0]}|\xi(s)|^2\Big)e^{C(t-t_0)}; \tag{8.4}$$

(ii) $\int_0^{t-t_0}[|b(t_0+s,X_{t_0+s})| + |\sigma(t_0+s,X_{t_0+s})|^2]ds < +\infty$ *a.s.;*

(iii) $X(t) - X(t_0) = \int_0^{t-t_0} b(t_0+s,X_{t_0+s})ds + \sigma(t_0+s,X_{t_0+s})dW(s)$.

We denote $\mathbb{P}_{(t_0,\xi)}$ the law of this solution.

Proofs of the uniqueness and existence, as well as the useful bound (8.4), can be found in [62].

Topological support of a stochastic process. In the statement of the link between a diffusion and the associated Kolmogorov equation, the domain of the PDE is related to the support of the random variable $X(t)$. In the same way, the support of the law (on the space of paths) of the process plays a key role in the path dependent extension of the Kolmogorov equation.

Recall that the *topological support* of a random variable X with values in a metric space is the smallest closed set $\mathrm{supp}(X)$ such that $\mathbb{P}(X \in \mathrm{supp}(X)) = 1$.

Viewing a process X with continuous paths as a random variable on $(C^0([0,T],\mathbb{R}^d), \|\cdot\|_\infty)$ leads to the notion of *topological support* of a continuous process.

The support may be characterized by the following property: it is the set $\mathrm{supp}(X)$ of paths $\omega \in C^0([0,T],\mathbb{R}^d)$ for which every Borel neighborhood has strictly positive measure, i.e.,

$$\mathrm{supp}(X) = \big\{\omega \in C^0\big([0,T],\mathbb{R}^d\big) \mid \forall \text{ Borel neigh. } V \text{ of } \omega,\ \mathbb{P}(X_T \in V) > 0\big\}.$$

Example 8.1.2. If W is a d-dimensional Wiener process with non-singular covariance matrix, then,

$$\mathrm{supp}(W) = \big\{\omega \in C^0\big([0,T],\mathbb{R}^d\big), \omega(0) = 0\big\}.$$

Example 8.1.3. For an Itô process

$$X(t,\omega) = x + \int_0^t \sigma(t,\omega)dW,$$

if $\mathbb{P}(\sigma(t,\omega)\cdot^{\mathrm{t}}\sigma(t,\omega) \ge \epsilon\, Id) = 1$, then $\mathrm{supp}(X) = \big\{\omega \in C^0\big([0,T],\mathbb{R}^d\big), \omega(0) = x\big\}$.

A classical result due to Stroock–Varadhan [68] characterizes the topological support of a multi-dimensional diffusion process (for a simple proof see Millet and Sanz-Solé [54]).

Example 8.1.4 (Stroock–Varadhan support theorem)**.** Let

$$b\colon [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad \sigma\colon [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$$

be Lipschitz continuous in x, uniformly on $[0,T]$. Then the law of the diffusion process

$$dX(t) = b(t,X(t))dt + \sigma(t,X(t))dW(t), \quad X(0)=x,$$

has a support given by the closure in $(C^0([0,T],\mathbb{R}^d),\|.\|_\infty)$ of the 'skeleton'

$$x + \left\{\omega\in H^1([0,T],\mathbb{R}^d), \exists h\in H^1([0,T],\mathbb{R}^d),\, \dot\omega(t) = b(t,\omega(t)) + \sigma(t,\omega(t))\dot h(t)\right\},$$

defined as the set of all solutions of the system of ODEs obtained when substituting $h\in H^1([0,T],\mathbb{R}^d)$ for W in the SDE.

We will denote by $\Lambda(T,X)$ the set of all stopped paths obtained from paths in $\mathrm{supp}(X)$:

$$\Lambda(T,X) = \left\{(t,\omega)\in\Lambda_T,\, \omega\in\mathrm{supp}(X)\right\}. \tag{8.5}$$

For a continuous process X, $\Lambda(T,X)\subset\mathcal{W}_T$.

8.1.2 Local martingales and harmonic functionals

Functionals of a process X which have the local martingale property play an important role in stochastic control theory [17] and mathematical finance. By analogy with the Markovian case, we call such functionals $\mathbb{P}$-harmonic functionals:

Definition 8.1.5 ($\mathbb{P}$-harmonic functionals)**.** *A non-anticipative functional F is called $\mathbb{P}$-harmonic if $Y(t)=F(t,X_t)$ is a $\mathbb{P}$-local martingale.*

The following result characterizes smooth $\mathbb{P}$-harmonic functionals as solutions to a *functional Kolmogorov equation*, see [10].

Theorem 8.1.6 (Functional equation for $\mathcal{C}^{1,2}$ martingales)**.** *If $F\in\mathbb{C}_b^{1,2}(\mathcal{W}_T)$ and $\mathcal{D}F\in\mathbb{C}_l^{0,0}$, then $Y(t)=F(t,X_t)$ is a local martingale if and only if F satisfies*

$$\mathcal{D}F(t,\omega_t) + b(t,\omega_t)\nabla_\omega F(t,\omega_t) + \frac{1}{2}\mathrm{tr}\left[\nabla_\omega^2 F(t,\omega)\sigma^{\mathrm{t}}\sigma(t,\omega)\right] = 0 \tag{8.6}$$

on the topological support of X in $(C^0([0,T],\mathbb{R}^d),\|.\|_\infty)$.

Proof. If $F\in\mathbb{C}_b^{1,2}(\mathcal{W}_T)$ is a solution of (8.6), the functional Itô formula (Theorem 6.2.3) applied to $Y(t)=F(t,X_t)$ shows that Y has the semimartingale decomposition

$$Y(t) = Y(0) + \int_0^t Z(u)du + \int_0^t \nabla_\omega F(u,X_u)\sigma(u,X_u)dW(u),$$

where

$$Z(t) = \mathcal{D}F(t, X_t) + b(t, X_t)\nabla_\omega F(t, X_t) + \frac{1}{2}\mathrm{tr}\big(\nabla_\omega^2 F(t, X_t)\sigma^{\mathrm{t}}\sigma(t, X_t)\big).$$

Equation (8.6) then implies that the finite variation term in (6.2) is almost surely zero. So, $Y(t) = Y(0) + \int_0^t \nabla_\omega F(u, X_u).\sigma(u, X_u)dW(u)$ and thus Y is a continuous local martingale.

Conversely, assume that $Y = F(X)$ is a local martingale. Let $A(t,\omega) = \sigma(t,\omega)^t\sigma(t,\omega)$. Then, by Lemma 5.1.5, Y is left-continuous. Suppose (8.6) is not satisfied for some $\omega \in \mathrm{supp}(X) \subset C^0([0,T], \mathbb{R}^d)$ and some $t_0 < T$. Then, there exist $\eta > 0$ and $\epsilon > 0$ such that

$$\left|\mathcal{D}F(t,\omega) + b(t,\omega)\nabla_\omega F(t,\omega) + \frac{1}{2}\mathrm{tr}\big[\nabla_\omega^2 F(t,\omega)A(t,\omega)\big]\right| > \epsilon,$$

for $t \in [t_0 - \eta, t_0]$, by left-continuity of the expression. Also, there exist a neighborhood V of ω in $C^0([0,T], \mathbb{R}^d), \|.\|_\infty)$ such that, for all $\omega' \in V$ and $t \in [t_0 - \eta, t_0]$,

$$\left|\mathcal{D}F(t,\omega') + b(t,\omega')\nabla_\omega F(t,\omega') + \frac{1}{2}\mathrm{tr}\big[\nabla_\omega^2 F(t,\omega')A(t,\omega')\big]\right| > \frac{\epsilon}{2}.$$

Since $\omega \in \mathrm{supp}\ (X)$, $\mathbb{P}(X \in V) > 0$, and so

$$\left\{(t,\omega) \in \mathcal{W}_T,\ \left|\mathcal{D}F(t,\omega) + b(t,\omega)\nabla_\omega F(t,\omega) + \frac{1}{2}\mathrm{tr}\big(\nabla_\omega^2 F(t,\omega)A(t,\omega)\big)\right| > \frac{\epsilon}{2}\right\}$$

has non-zero measure with respect to $dt \times d\mathbb{P}$. Applying the functional Itô formula (6.2) to the process $Y(t) = F(t, X_t)$ then leads to a contradiction because, as a continuous local martingale, its finite variation component should be zero $dt \times d\mathbb{P}$-a.e. □

In the case where $F(t,\omega) = f(t,\omega(t))$ and the coefficients b and σ are not path dependent, this equation reduces to the well-known backward Kolmogorov equation [46].

Remark 8.1.7 (Relation with infinite-dimensional Kolmogorov equations in Hilbert spaces)**.** Note that the second-order operator ∇_ω^2 is an iterated directional derivative, and not a second-order Fréchet or "H-derivative" so, this equation is different from the infinite-dimensional Kolmogorov equations in Hilbert spaces as described, for example, by Da Prato and Zabczyk [16]. The relation between these two classes of infinite-dimensional Kolmogorov equations has been studied recently by Flandoli and Zanco [28], who show that, although one can partly recover some results for (8.6) using the theory of infinite-dimensional Kolmogorov equations in Hilbert spaces, this can only be done at the price of higher regularity requirements (not the least of them being Fréchet differentiability), which exclude many interesting examples.

When $X = W$ is a d-dimensional Wiener process, Theorem 8.1.6 gives a characterization of regular Brownian local martingales as solutions of a 'functional heat equation'.

Corollary 8.1.8 (Harmonic functionals on $\mathcal{W}_T$)**.** *Let $F \in \mathbb{C}_b^{1,2}(\mathcal{W}_T)$. Then, $(Y(t) = F(t, W_t), t \in [0,T])$ is a local martingale if and only if for every $t \in [0,T]$ and every $\omega \in C^0\big([0,T], \mathbb{R}^d\big)$,*

$$\mathcal{D}F(t,\omega) + \frac{1}{2}\,\mathrm{tr}\big(\nabla_\omega^2 F(t,\omega)\big) = 0.$$

8.1.3 Sub-solutions and super-solutions

By analogy with the case of finite-dimensional parabolic PDEs, we introduce the notion of sub- and super-solution to (8.6).

Definition 8.1.9 (Sub-solutions and super-solutions)**.** $F \in \mathbb{C}^{1,2}(\Lambda_T)$ *is called a* super-solution *of* (8.6) *on a domain $U \subset \Lambda_T$ if, for all $(t,\omega) \in U$,*

$$\mathcal{D}F(t,\omega) + b(t,\omega)\cdot\nabla_\omega F(t,\omega) + \frac{1}{2}\mathrm{tr}\big[\nabla_\omega^2 F(t,\omega)\sigma^{\mathrm{t}}\sigma(t,\omega)\big] \leq 0. \qquad (8.7)$$

Simillarly, $F \in \mathbb{C}_{\mathrm{loc}}^{1,2}(\Lambda_T)$ is called a sub-solution *of* (8.6) *on U if, for all $(t,\omega) \in U$,*

$$\mathcal{D}F(t,\omega) + b(t,\omega)\cdot\nabla_\omega F(t,\omega) + \frac{1}{2}\mathrm{tr}\big[\nabla_\omega^2 F(t,\omega)\sigma^{\mathrm{t}}\sigma(t,\omega)\big] \geq 0. \qquad (8.8)$$

The following result shows that $\mathbb{P}$-integrable sub- and super-solutions have a natural connection with $\mathbb{P}$-submartingales and $\mathbb{P}$-supermartingales.

Proposition 8.1.10. *Let $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$ be such that, for all $t \in [0,T]$, we have $E(|F(t,X_t)|) < \infty$. Set $Y(t) = F(t,X_t)$.*

(i) *Y is a submartingale if and only if F is a sub-solution of* (8.6) *on* $\mathrm{supp}(X)$.

(ii) *Y is a supermartingale if and only if F is a super-solution of* (8.6) *on* $\mathrm{supp}(X)$.

Proof. If $F \in \mathbb{C}^{1,2}$ is a super-solution of (8.6), the application of the Functional Itô formula (Theorem 6.2.3, Eq. (6.2)) gives

$$Y(t) - Y(0) = \int_0^t \nabla_\omega F(u,X_u)\sigma(u,X_u)dW(u)$$

$$+ \int_0^t \left(\mathcal{D}F(u,X_u) + \nabla_\omega F(u,X_u)b(u,X_u) + \frac{1}{2}\mathrm{tr}\big(\nabla_\omega^2 F(u,X_u)A(u,X_u)\big)\right) du \text{ a.s.},$$

where the first term is a local martingale and, since F is a super-solution, the finite variation term (second line) is a decreasing process, so Y is a supermartingale.

Conversely, let $F \in \mathbb{C}^{1,2}(\Lambda_T)$ be such that $Y(t) = F(t,X_t)$ is a supermartingale. Let $Y = M - H$ be its Doob–Meyer decomposition, where M is a local martingale and H an increasing process. By the functional Itô formula, Y has the above decomposition so it is a continuous supermartingale. By the uniqueness of the Doob–Meyer decomposition, the second term is thus a decreasing process so,

$$\mathcal{D}F(t,X_t)+\nabla_\omega F(t,X_t)b(t,X_t)+\frac{1}{2}\mathrm{tr}\big(\nabla_\omega^2 F(t,X_t)A(t,X_t)\big)\le 0 \ dt\times d\mathbb{P}\text{-a.e.}$$

This implies that the set

$$S=\big\{\omega\in C^0\big([0,T],\mathbb{R}^d\big)\mid (8.7)\text{ holds for all } t\in[0,T]\big\}$$

includes any Borel set $B\subset C^0([0,T],\mathbb{R}^d)$ with $\mathbb{P}(X_T\in B)>0$. Using continuity of paths,

$$S=\bigcap_{t\in[0,T]\cap\mathbb{Q}}\big\{\omega\in C^0\big([0,T],\mathbb{R}^d\big)\mid (8.7)\text{ holds for }(t,\omega)\big\}.$$

Since $F\in\mathbb{C}^{1,2}(\Lambda_T)$, this is a closed set in $(C^0([0,T],\mathbb{R}^d),\|\cdot\|_\infty)$ so, S contains the topological support of X, i.e., (8.7) holds on $\Lambda_T(X)$. □

8.1.4 Comparison principle and uniqueness

A key property of the functional Kolmogorov equation (8.6) is the comparison principle. As in the finite-dimensional case, this requires imposing an integrability condition.

Theorem 8.1.11 (Comparison principle)**.** *Let $\underline{F}\in\mathbb{C}^{1,2}(\Lambda_T)$ be a sub-solution of (8.6) and $\overline{F}\in\mathbb{C}^{1,2}(\Lambda_T)$ be a super-solution of (8.6) such that, for every $\omega\in C^0\big([0,T],\mathbb{R}^d\big)$, $\underline{F}(T,\omega)\le\overline{F}(T,\omega)$, and*

$$E\Big(\sup_{t\in[0,T]}|\underline{F}(t,X_t)-\overline{F}(t,X_t)|\Big)<\infty.$$

Then, $\forall t\in[0,T],\forall\omega\in\mathrm{supp}(X)$,

$$\underline{F}(t,\omega)\le\overline{F}(t,\omega). \tag{8.9}$$

Proof. Since $F=\underline{F}-\overline{F}\in\mathbb{C}^{1,2}(\Lambda_T)$ is a sub-solution of (8.6), by Proposition 8.1.10, the process Y defined by $Y(t)=F(t,X_t)$ is a submartingale. Since $Y(T)=\underline{F}(T,X_T)-\overline{F}(T,X_T)\le 0$, we have

$$\forall t\in[0,T[,\quad Y(t)\le 0\quad \mathbb{P}\text{-a.s.}$$

Define

$$S=\big\{\omega\in C^0([0,T],\mathbb{R}^d),\ \forall t\in[0,T],\ \underline{F}(t,\omega)\le\overline{F}(t,\omega)\big\}.$$

Using continuity of paths, and since $F\in\mathbb{C}^{0,0}(\mathcal{W}_T)$,

$$S=\bigcap_{t\in\mathbb{Q}\cap[0,T]}\big\{\omega\in\mathrm{supp}(X),\ \underline{F}(t,\omega)\le\overline{F}(t,\omega)\big\}$$

is closed in $(C^0([0,T],\mathbb{R}^d),\|.\|_\infty)$. If (8.9) does not hold, then there exist $t_0 \in [0,T[$ and $\omega \in \operatorname{supp}(X)$ such that $F(t_0,\omega) > 0$. Since $O = \operatorname{supp}(X) - S$ is a non-empty open subset of $\operatorname{supp}(X)$, there exists an open set $A \subset O \subset \operatorname{supp}(X)$ containing ω and $h > 0$ such that, for every $t \in [t_0 - h, t]$ and $\omega \in A$, $F(t,\omega) > 0$. But $\mathbb{P}(X \in A) > 0$ implies that $\int dt \times d\mathbb{P}\, 1_{F(t,\omega)>0} > 0$, which contradicts the above assumption. □

The following uniqueness result for $\mathbb{P}$-uniformly integrable solutions of the Functional Kolmogorov equation (8.6) is a straightforward consequence of the comparison principle.

Theorem 8.1.12 (Uniqueness of solutions). *Let $H\colon (C^0([0,T],\mathbb{R}^d),\|\cdot\|_\infty) \to \mathbb{R}$ be a continuous functional, and let $F^1, F^2 \in \mathbb{C}_b^{1,2}(\mathcal{W}_T)$ be solutions of the Functional Kolmogorov equation* (8.6) *verifying*

$$F^1(T,\omega) = F^2(T,\omega) = H(\omega) \tag{8.10}$$

and

$$E\Big[\sup_{t\in[0,T]} \big|F^1(t,X_t) - F^2(t,X_t)\big|\Big] < \infty. \tag{8.11}$$

Then, they coincide on the topological support of X, i.e.,

$$\forall \omega \in \operatorname{supp}(X), \quad \forall t \in [0,T], \quad F^1(t,\omega) = F^2(t,\omega).$$

The integrability condition in this result (or some variation of it) is required for uniqueness: indeed, even in the finite-dimensional setting of a heat equation in $\mathbb{R}^d$, one cannot do without an integrability condition with respect to the heat kernel. Without this condition, we can only assert that $F(t,X_t)$ is a local martingale, so it is not uniquely determined by its terminal value and counterexamples to uniqueness are easy to construct.

Note also that uniqueness holds on the support of X, the process associated with the (path dependent) operator appearing in the Kolmogorov equation. The local martingale property of $F(X)$ implies no restriction on the behavior of F outside $\operatorname{supp}(X)$ so, there is no reason to expect uniqueness or comparison of solutions on the whole path space.

8.1.5 Feynman–Kac formula for path dependent functionals

As observed above, any $\mathbb{P}$-uniformly integrable solution F of the functional Kolmogorov equation yields a $\mathbb{P}$-martingale $Y(t) = F(t,X_t)$. Combined with the uniqueness result (Theorem 8.1.12), this leads to an extension of the well-known Feynman–Kac representation to the path dependent case.

Theorem 8.1.13 (A Feynman–Kac formula for path dependent functionals). *Let $H\colon (C^0([0,T]),\|\cdot\|_\infty) \to \mathbb{R}$ be continuous. If, for every $\omega \in C^0([0,T])$, $F \in \mathbb{C}_b^{1,2}(\mathcal{W}_T)$ verifies*

$$\mathcal{D}F(t,\omega) + b(t,\omega)\cdot\nabla_\omega F(t,\omega) + \frac{1}{2}\operatorname{tr}\big[\nabla_\omega^2 F(t,\omega)\sigma^{\mathrm{t}}\sigma(t,\omega)\big] = 0,$$

$$F(T,\omega) = H(\omega), \quad E\Big[\sup_{t\in[0,T]} |F(t,X_t)|\Big] < \infty,$$

then F has the probabilistic representation

$$\forall \omega \in \mathrm{supp}(X), \quad F(t,\omega) = E^{\mathbb{P}}\big[H(X_T)\,|\,X_t = \omega_t\big] = E^{\mathbb{P}_{(t,\omega)}}[H(X_T)],$$

where $\mathbb{P}_{(t,\omega)}$ is the law of the unique solution of the SDE $dX(t) = b(t,X_t)dt + \sigma(t,X_t)dW(t)$ *with initial condition $X_t = \omega_t$. In particular,*

$$F(t,X_t) = E^{\mathbb{P}}\big[H(X_T)\,|\,\mathcal{F}_t\big] \quad dt \times d\mathbb{P}\text{-}a.s.$$

8.2 FBSDEs and semilinear functional PDEs

The relation between stochastic processes and partial differential equations extends to the semilinear case: in the Markovian setting, there is a well-known relation between semilinear PDEs of parabolic type and Markovian forward-backward stochastic differential equations (FBSDE), see [24, 58, 74]. The tools of Functional Itô Calculus allows to extend the relation between semilinear PDEs and FBSDEs beyond the Markovian case, to non-Markovian FBSDEs with path dependent coefficients.

Consider the forward-backward stochastic differential equation with path dependent coefficients

$$X(t) = X(0) + \int_0^t b\big(u, X_{u-}, X(u)\big)du + \int_0^t \sigma\big(u, X_{u-}, X(u)\big)dW(u), \tag{8.12}$$

$$Y(t) = H(X_T) + \int_t^T f\big(s, X_{s-}, X(s), Y(s), Z(s)\big)ds - \int_t^T Z dM, \tag{8.13}$$

whose coefficients

$$b\colon \mathcal{W}_T \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad \sigma\colon \mathcal{W}_T \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}, \quad f\colon \mathcal{W}_T \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

are assumed to verify Assumption 7.7.1. Under these assumptions,

(i) the forward SDE given by (8.12) has a unique strong solution X with $E(\sup_{t\in[0,T]} |X(t)|^2) < \infty$, whose martingale part we denote by M, and

(ii) the FBSDE (8.12)–(8.13) admits a unique solution $(Y,Z) \in \mathcal{S}^{1,2}(M) \times \mathcal{L}^2(M)$ with $E(\sup_{t\in[0,T]} |Y(t)|^2) < \infty$.

Define

$$B(t,\omega) = b\big(t, \omega_{t-}, \omega(t)\big), \quad A(t,\omega) = \sigma\big(t, \omega_{t-}, \omega(t)\big)^{\mathrm{t}} \sigma\big(t, \omega_{t-}, \omega(t)\big).$$

In the 'Markovian' case, where the coefficients are not path dependent, the solution of the FBSDE can be obtained by solving a semilinear PDE: if $f \in C^{1,2}([0,T) \times \mathbb{R}^d])$ is a solution of

$$\partial_t f(t,\omega) + f\big(t, x, f(t,x), \nabla f(t,x)\big) + b(t,x)\nabla f(t,x) + \frac{1}{2}\mathrm{tr}\big[a(t,\omega)\cdot\nabla^2 f(t,x)\big] = 0,$$

then a solution of the FBSDE is given by $(Y(t), Z(t)) = (f(t, X(t)), \nabla f(t, X(t)))$. Here, the function u 'decouples' the solution of the backward equation from the solution of the forward equation. In the general path dependent case (8.12)–(8.13) we have a similar object, named the 'decoupling random field' in [51] following earlier ideas from [50].

The following result shows that such a 'decoupling random field' for the path dependent FBSDE (8.12)–(8.13) may be constructed as a solution of a semilinear functional PDE, analogously to the Markovian case.

Theorem 8.2.1 (FBSDEs as semilinear path dependent PDEs)**.** *Let $F \in \mathbb{C}^{1,2}_{\mathrm{loc}}(\mathcal{W}_T)$ be a solution of the path dependent semilinear equation*

$$\begin{aligned}\mathcal{D}F(t,\omega) &+ f\big(t, \omega_{t-}, \omega(t), F(t,\omega), \nabla_\omega F(t,\omega)\big)\\ &+ B(t,\omega)\cdot\nabla_\omega F(t,\omega) + \frac{1}{2}\mathrm{tr}\big[A(t,\omega).\nabla^2_\omega F(t,\omega)\big] = 0,\end{aligned}$$

for $\omega \in \mathrm{supp}(X)$, $t \in [0,T]$, and with $F(T,\omega) = H(\omega)$. Then, the pair of processes (Y, Z) given by

$$\big(Y(t), Z(t)\big) = \big(F(t, X_t), \nabla_\omega F(t, X_t)\big)$$

is a solution of the FBSDE (8.12)–(8.13)*.*

Remark 8.2.2. In particular, under the conditions of Theorem 8.2.1 the random field $u\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ defined by

$$u(t,x,\omega) = F(t, X_{t-}(t,\omega) + x1_{[t,T]})$$

is a 'decoupling random field' in the sense of [51].

Proof of Theorem 8.2.1. Let $F \in \mathbb{C}^{1,2}_{\mathrm{loc}}(\mathcal{W}_T)$ be a solution of the semilinear equation above. Then the processes (Y, Z) defined above are $\mathbb{F}$-adapted. Applying the functional Itô formula to $F(t, X_t)$ yields

$$\begin{aligned}H(X_T) - F(t, X_t) &= \int_t^T \nabla_\omega F(u, X_u) dM\\ &+ \int_t^T \Big(\mathcal{D}F(u, X_u) + B(u, X_u)\nabla_\omega F(u, X_u) + \frac{1}{2}\mathrm{tr}\big[A(u, X_u)\nabla^2_\omega F(u, X_u)\big]\Big) du.\end{aligned}$$

Since F is a solution of the equation, the term in the last integral is equal to

$$-f(t, X_{t-}, X(t), F(t, X_t), \nabla_\omega F(t, X_t)) = -f(t, X_{t-}, X(t), Y(t), Z(t)).$$

So, $(Y(t), Z(t)) = (F(t, X_t), \nabla_\omega F(t, X_t))$ verifies

$$Y(t) = H(X_T) - \int_t^T f\big(s, X_{s-}, X(s), Y(s), Z(s)\big)ds - \int_t^T ZdM.$$

Therefore, (Y, Z) is a solution to the FBSDE (8.12)–(8.13). □

This result provides the "Hamiltonian" version of the FBSDE (8.12)–(8.13), in the terminology of [74].

8.3 Non-Markovian stochastic control and path dependent HJB equations

An interesting application of the Functional Itô Calculus is the study of non-Markovian stochastic control problems, where both the evolution of the state of the system and the control policy are allowed to be path dependent.

Non-Markovian stochastic control problems were studied by Peng [59], who showed that, when the characteristics of the controlled process are allowed to be stochastic (i.e., path dependent), the optimality conditions can be expressed in terms of a stochastic Hamilton–Jacobi–Bellman equation. The relation with FBSDEs is explored systematically in the monograph [74].

We formulate a mathematical framework for stochastic optimal control where the evolution of the state variable, the control policy and the objective are allowed to be non-anticipative path dependent functionals, and show how the Functional Itô formula in Theorem 6.2.3 characterizes the value functional as the solution to a *functional Hamilton–Jacobi–Bellman equation*.

This section is based on [32].

Let W be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the $\mathbb{P}$-augmented natural filtration of W.

Let C be a subset of $\mathbb{R}^m$. We consider the controlled stochastic differential equation

$$dX(t) = b\big(t, X_t, \alpha(t)\big)dt + \sigma\big(t, X_t, \alpha(t)\big)dW(t), \tag{8.14}$$

where the coefficients

$$b\colon \mathcal{W}_T \times C \longrightarrow \mathbb{R}^d, \quad \sigma\colon \mathcal{W}_T \times C \longrightarrow \mathbb{R}^{d\times d}$$

are allowed to be path dependent and assumed to verify the conditions of Proposition 8.1.1 uniformly with respect to $\alpha \in C$, and the control α belongs to the set $\mathcal{A}(\mathbb{F})$ of $\mathbb{F}$-progressively measurable processes with values in C verifying

$$E\bigg(\int_0^T \|\alpha(t)\|^2 dt\bigg) < \infty.$$

Then, for any initial condition $(t,\omega) \in \mathcal{W}_T$ and for each $\alpha \in \mathcal{A}(\mathbb{F})$, the stochastic differential equation (8.14) admits a unique strong solution, which we denote by $X^{(t,\omega,\alpha)}$.

Let $g\colon (C^0([0,T],\mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$ be a continuous map representing a terminal cost and $L\colon \mathcal{W}_T \times C \to L(t,\omega,u)$ a non-anticipative functional representing a possibly path dependent running cost. We assume

(i) $\exists K > 0$ such that $-K \le g(\omega) \le K(1 + \sup_{s\in[0,t]} |\omega(s)|^2)$;

(ii) $\exists K' > 0$ such that $-K' \le L(t,\omega,u) \le K'(1 + \sup_{s\in[0,t]} |\omega(s)|^2)$.

Then, thanks to (8.4), the cost functional

$$J(t,\omega,\alpha) = E\left[g\bigl(X_T^{(t,\omega,\alpha)}\bigr) + \int_t^T L\bigl(s, X_s^{(t,\omega,\alpha)}, \alpha(s)\bigr)ds\right] < \infty$$

for $(t,\omega,\alpha) \in \mathcal{W}_T \times \mathcal{A}$ defines a non-anticipative map $J\colon \mathcal{W}_T \times \mathcal{A}(\mathbb{F}) \to \mathbb{R}$.

We consider the optimal control problem

$$\inf_{\alpha\in\mathcal{A}(\mathbb{F})} J(t,\omega,\alpha)$$

whose value functional is denoted, for $(t,\omega) \in \mathcal{W}_T$,

$$V(t,\omega) = \inf_{\alpha\in\mathcal{A}(\mathbb{F})} J(t,\omega,\alpha). \tag{8.15}$$

From the above assumptions and the estimate (8.4), $V(t,\omega)$ verifies

$$\exists K'' > 0 \text{ such that } -K'' \le V(t,\omega) \le K''\Bigl(1 + \sup_{s\in[0,t]} |\omega(s)|^2\Bigr).$$

Introduce now the *Hamiltonian* associated to the control problem [74] as the non-anticipative functional $H\colon \mathcal{W}_T \times \mathbb{R}^d \times S_d^+ \to \mathbb{R}$ given by

$$H(t,\omega,\rho,A) = \inf_{u\in C}\left\{\frac{1}{2}\mathrm{tr}\bigl(A\sigma(t,\omega,u)^{\mathrm{t}}\sigma(t,\omega,u)\bigr) + b(t,\omega,u)\rho + L(t,\omega,u)\right\}.$$

It is readily observed that, for any $\alpha \in \mathcal{A}(\mathbb{F})$, the process

$$V\bigl(t, X_t^{(t_0,\omega,\alpha)}\bigr) + \int_{t_0}^t L\bigl(s, X_s^{(t_0,\omega,\alpha)}, \alpha(s)\bigr)ds$$

has the submartingale property. The martingale optimality principle [17] then characterizes an optimal control α^* as one for which this process is a martingale.

We can now use the Functional Itô Formula (Theorem 6.2.3) to give a sufficient condition for a functional U to be equal to the value functional V and for a control α^* to be optimal. This condition takes the form of a path dependent Hamilton–Jacobi–Bellman equation.

Theorem 8.3.1 (Verification Theorem)**.** *Let $U \in \mathbb{C}_b^{1,2}(\mathcal{W}_T)$ be a solution of the functional* HJB *equation,*

$$\forall (t,\omega) \in \mathcal{W}_T, \quad \mathcal{D}U\big(t,\omega\big) + H\big(t,\omega,\nabla_\omega U(t,\omega),\nabla_\omega^2 U(t,\omega)\big) = 0, \tag{8.16}$$

satisfying $U(T,\omega) = g(\omega)$ and the quadratic growth condition

$$|U(t,\omega)| \le C \sup_{s\le t} |\omega(s)|^2. \tag{8.17}$$

Then, for any $\omega \in C^0([0,T],\mathbb{R}^d)$ and any admissible control $\alpha \in \mathcal{A}(\mathbb{F})$,

$$U(t,\omega) \le J(t,\omega,\alpha). \tag{8.18}$$

If, furthermore, for $\omega \in C^0([0,T],\mathbb{R}^d)$ there exists $\alpha^* \in \mathcal{A}(\mathbb{F})$ such that

$$\begin{aligned} &H\big(s, X_s^{(t,\omega,\alpha^*)}, \nabla_\omega U\big(s, X_s^{(t,\omega,\alpha^*)}\big), \nabla_\omega^2 U\big(s, X_s^{(t,\omega,\alpha^*)}\big)\big) \\ &= \frac{1}{2}\mathrm{tr}\big[\nabla_\omega^2 U\big(s, X_s^{(t,\omega,\alpha^*)}\big)\sigma\big(s, X_s^{(t,\omega,\alpha^*)}, \alpha^*(s)\big)^{\mathrm{t}}\sigma\big(s, X_s^{(t,\omega,\alpha^*)}, \alpha^*(s)\big)\big] \\ &\quad + \nabla_\omega U\big(s, X_s^{(t,\omega,\alpha^*)}\big) b\big(s, X_s^{(t,\omega,\alpha^*)}, \alpha^*(s)\big) + L\big(s, X_s^{(t,\omega,\alpha^*)}, \alpha^*(s)\big) \end{aligned} \tag{8.19}$$

$ds \times d\mathbb{P}$-a.e. for $s \in [t,T]$, then $U(t,\omega) = V(t,\omega) = J(t,\omega,\alpha^*)$ and α^* is an optimal control.

Proof. Let $\alpha \in \mathcal{A}(\mathbb{F})$ be an admissible control, $t < T$ and $\omega \in C^0([0,T],\mathbb{R}^d)$. Applying the Functional Itô Formula yields, for $s \in [t,T]$,

$$\begin{aligned} &U\big(s, X_s^{(t,\omega,\alpha)}\big) - U(t,\omega) \\ &= \int_t^s \nabla_\omega U\big(u, X_u^{(t,\omega,\alpha)}\big)\sigma\big(u, X_u^{(t,\omega,\alpha)}, \alpha(u)\big) dW(u) \\ &\quad + \int_t^s \mathcal{D}U\big(u, X_u^{(t,\omega,\alpha)}\big) + \nabla_\omega U\big(u, X_u^{(t,\omega,\alpha)}\big) b\big(u, X_u^{(t,\omega,\alpha)}, \alpha(u)\big) du \\ &\quad + \int_t^s \frac{1}{2}\mathrm{tr}\Big(\nabla_\omega^2 U\big(u, X_u^{(t,\omega,\alpha)}\big)\sigma^{\mathrm{t}}\sigma\big(u, X_u^{(t,\omega,\alpha)}\big)\Big) du. \end{aligned}$$

Since U verifies the functional Hamilton–Jacobi–Bellman equation, we have

$$\begin{aligned} U\big(s, X_s^{(t,\omega,\alpha)}\big) - U(t,\omega) \ge & \int_t^s \nabla_\omega U\big(u, X_u^{(t,\omega,\alpha)}\big)\sigma\big(u, X_u^{(t,\omega,\alpha)}, \alpha(u)\big) dW(u) \\ & - \int_t^s L\big(u, X_u^{(t,\omega,\alpha)}, \alpha(u)\big) du. \end{aligned}$$

In other words, $U(s, X_s^{(t,\omega,\alpha)}) - U(t,\omega) + \int_t^s L(u, X_u^{(t,\omega,\alpha)}, \alpha(u))du$ is a local submartingale. The estimate (8.17) and the L^2 estimate (8.4) guarantee that it is actually a submartingale, hence, taking $s \to T$, the left continuity of U yields

$$E\bigg[g\big(X_T^{(t,\omega,\alpha)}\big) + \int_0^{T-t} L\big(t+u, X_{t+u}^{(t,\omega,\alpha)}, \alpha(u)\big) du\bigg] \ge U(t,\omega).$$

This being true for any $\alpha \in \mathcal{A}(\mathbb{F})$, we conclude that $U(t, \omega) \leq J(t, \omega, \alpha)$.

Assume now that $\alpha^* \in \mathcal{A}(\mathbb{F})$ verifies (8.19). Taking $\alpha = \alpha^*$ transforms inequalities to equalities, submartingale to martingale, and hence establishes the second part of the theorem. □

This proof can be adapted [32] to the case where all coefficients, except σ, depend on the quadratic variation of X as well, using the approach outlined in Section 6.3. The subtle point is that if σ depends on the control, the Functional Itô Formula (Theorem 6.2.3) does not apply to $U(s, X_s^\alpha, [X^\alpha]_s)$ because $d[X^\alpha](s)/ds$ would not necessarily admit a right continuous representative.

8.4 Weak solutions

The above results are 'verification theorems' which, assuming the existence of a solution to the Functional Kolmogorov Equation with a given regularity, derive a probabilistic property or probabilistic representation of this solution. The key issue when applying such results is to be able to prove the regularity conditions needed to apply the Functional Itô Formula. In the case of linear Kolmogorov equations, this is equivalent to constructing, given a functional H, a *smooth* version of the conditional expectation $E[H|\mathcal{F}_t]$, i.e., a functional representation admitting vertical and horizontal derivatives.

As with the case of finite-dimensional PDEs, such *strong solutions* —with the required differentiability— may fail to exist in many examples of interest and, even if they exist, proving pathwise regularity is not easy in general and requires imposing many smoothness conditions on the terminal functional H, due to the absence of any regularizing effect as in the finite-dimensional parabolic case [13, 61].

A more general approach is provided by the notion of *weak solution*, which we now define, using the tools developed in Chapter 7. Consider, as above, a semimartingale X defined as the solution of a stochastic differential equation

$$X(t) = X(0) + \int_0^t b(u, X_u)du + \int_0^t \sigma(u, X_u)dW(u) = X(0) + \int_0^t b(u, X_u)du + M(t)$$

whose coefficients b, σ are non-anticipative path dependent functionals, verifying the assumptions of Proposition 8.1.1. We assume that M, the martingale part of X, is a square integrable martingale.

We can then use the notion of weak derivative ∇_M on the Sobolev space $\mathcal{S}^{1,2}(M)$ of semimartingales introduced in Section 7.5.

The idea is to introduce a probability measure $\mathbb{P}$ on $D([0,T], \mathbb{R}^d)$, under which X is a semimartingale, and requiring (8.6) to hold $dt \times d\mathbb{P}$-a.e.

For a classical solution $F \in \mathbb{C}_b^{1,2}(\mathcal{W}_T)$, requiring (8.6) to hold on the support of M is equivalent to requiring

$$\mathcal{A}F(t,X_t)=\mathcal{D}F(t,X_t)+\nabla_\omega F(t,X_t)b(t,X_t)+\frac{1}{2}\mathrm{tr}\big[\nabla_\omega^2 F(t,X_t)\sigma^{\mathrm{t}}\sigma(t,X_t)\big]=0, \tag{8.20}$$

$dt\times d\mathbb{P}$-a.e.

Let $\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})$ be the space of $\mathbb{F}$-adapted processes $\phi\colon[0,T]\times\Omega\to\mathbb{R}$ with

$$E\bigg(\int_0^T |\phi(t)|^2dt\bigg)<\infty,$$

and $\mathcal{A}^2(\mathbb{F})$ the space of absolutely continuous processes whose Radon–Nikodým derivative lies in $\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})$:

$$\mathcal{A}^2(\mathbb{F})=\bigg\{\Phi\in\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})\mid \frac{d\Phi}{dt}\in\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})\bigg\}.$$

Using the density of $\mathcal{A}^2(\mathbb{F})$ in $\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})$, (8.20) is equivalent to requiring

$$\forall\Phi\in\mathcal{A}^2(\mathbb{F}),\quad E\bigg(\int_0^T\Phi(t)\mathcal{A}F(t,X_t)dt\bigg)=0, \tag{8.21}$$

where we can choose $\Phi(0)=0$ without loss of generality. But (8.21) is well-defined as soon as $\mathcal{A}F(X)\in\mathcal{L}^2(dt\times d\mathbb{P})$. This motivates the following definition.

Definition 8.4.1 (Sobolev space of non-anticipative functionals)**.** *We define $\mathbb{W}^{1,2}(\mathbb{P})$ as the space of (i.e., $dt\times d\mathbb{P}$-equivalence classes of) non-anticipative functionals $F\colon(\Lambda_T,d_\infty)\to\mathbb{R}$ such that the process $S=F(X)$ defined by $S(t)=F(t,X_t)$ belongs to $\mathcal{S}^{1,2}(M)$. Then, $S=F(X)$ has a semimartingale decomposition*

$$S(t)=F(t,X_t)=S(0)+\int_0^t\nabla_M S dM+H(t),\quad \textit{with}\quad \frac{dH}{dt}\in\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P}).$$

$\mathbb{W}^{1,2}(\mathbb{P})$ is a Hilbert space, equipped with the norm

$$\begin{aligned}\|F\|_{1,2}^2&=\|F(X)\|_{\mathcal{S}^{1,2}}^2\\&=E^{\mathbb{P}}\bigg(|F(0,X_0)|^2+\int_0^T\mathrm{tr}\big(\nabla_M F(X)\cdot^{\mathrm{t}}\nabla_M F(X)\cdot d[M]\big)+\int_0^T\bigg|\frac{dH}{dt}\bigg|^2dt\bigg).\end{aligned}$$

Denoting by $\mathbb{L}^2(\mathbb{P})$ the space of square integrable non-anticipative functionals, we have that for $F\in\mathbb{W}^{1,2}(\mathbb{P})$, its vertical derivative lies in $\mathbb{L}^2(\mathbb{P})$ and

$$\begin{aligned}\nabla_\omega\colon \mathbb{W}^{1,2}(\mathbb{P})&\longrightarrow\mathbb{L}^2(\mathbb{P}),\\ F&\longmapsto\nabla_\omega F\end{aligned}$$

is a continuous map.

Using linear combinations of smooth cylindrical functionals, one can show that this definition is equivalent to the following alternative construction.

Definition 8.4.2 ($\mathbb{W}^{1,2}(\mathbb{P})$: alternative definition)**.** *$\mathbb{W}^{1,2}(\mathbb{P})$ is the completion of $\mathbb{C}_{\mathrm{b}}^{1,2}(\mathcal{W}_T)$ with respect to the norm*

$$\|F\|_{1,2}^2 = E^{\mathbb{P}}\left(\left|F(0,X_0)\right|^2 + \int_0^T \operatorname{tr}\left(\nabla_\omega F(t,X_t)^{\mathrm{t}}\nabla_\omega F(t,X_t)d[M]\right)\right)$$
$$+ E^{\mathbb{P}}\left(\left|\int_0^T \left|\frac{d}{dt}\left(F(t,X_t) - \int_0^t \nabla_\omega F(u,X_u)dM\right)\right|^2 dt\right).$$

In particular, $\mathbb{C}_{\mathrm{loc}}^{1,2}(\mathcal{W}_T)\cap\mathbb{W}^{1,2}(\mathbb{P})$ is dense in $(\mathbb{W}^{1,2}(\mathbb{P}),\|.\|_{1,2})$.

For $F\in\mathbb{W}^{1,2}(\mathbb{P})$, let $S(t)=F(t,X_t)$ and M be the martingale part of X. The process $S(t)-\int_0^t \nabla_M S(u)\cdot dM$ has absolutely continuous paths and one can then define

$$\mathcal{A}F(t,X_t) := \frac{d}{dt}\left(F(t,X_t) - \int_0^t \nabla_\omega F(u,X_u).dM\right) \in \mathcal{L}^2\left(\mathbb{F}, dt\times d\mathbb{P}\right).$$

Remark 8.4.3. Note that, in general, it is not possible to define "$\mathcal{D}F(t,X_t)$" as a real-valued process for $F\in\mathbb{W}^{1,2}(\mathbb{P})$. As noted in Section 7.5, this requires $F\in\mathbb{W}^{2,2}(\mathbb{P})$.

Let U be the process defined by

$$U(t) = F(T,X_T) - F(t,X_t) - \int_t^T \nabla_M F(u,X_u)dM.$$

Then the paths of U lie in the Sobolev space $H^1([0,T],\mathbb{R})$,

$$\frac{dU}{dt} = -\mathcal{A}F(t,X_t), \quad \text{and} \quad U(T)=0.$$

Thus, we can apply the integration by parts formula for H^1 Sobolev functions [65] pathwise, and rewrite (8.21) as

$$\forall\Phi\in\mathcal{A}^2(\mathbb{F}), \quad \int_0^T dt\ \Phi(t)\ \frac{d}{dt}\left(F(t,X_t) - \int_0^t \nabla_M F(u,X_u)dM\right)$$
$$= \int_0^T dt\ \phi(t)\ \left(F(T,X_T) - F(t,X_t) - \int_t^T \nabla_M F(u,X_u)dM\right).$$

We are now ready to formulate the notion of weak solution in $\mathbb{W}^{1,2}(\mathbb{P})$.

Definition 8.4.4 (Weak solution)**.** *$F\in\mathbb{W}^{1,2}(\mathbb{P})$ is said to be a* weak solution *of*

$$\mathcal{D}F(t,\omega) + b(t,\omega)\nabla_\omega F(t,\omega) + \frac{1}{2}\operatorname{tr}\left[\nabla_\omega^2 F(t,\omega)\sigma^{\mathrm{t}}\sigma(t,\omega)\right] = 0$$

on supp(X) *with terminal condition* $H(\cdot) \in L^2(\mathcal{F}_T, \mathbb{P})$ *if* $F(T,\cdot) = H(\cdot)$ *and, for every* $\phi \in \mathcal{L}^2(\mathbb{F}, dt \times d\mathbb{P})$,

$$E\left[\int_0^T dt\, \phi(t)\left(H(X_T) - F(t, X_t) - \int_t^T \nabla_M F(u, X_u) dM\right)\right] = 0. \tag{8.22}$$

The preceding discussion shows that any square integrable classical solution $F \in \mathbb{C}^{1,2}_{\mathrm{loc}}(\mathcal{W}_T) \cap \mathcal{L}^2(dt \times d\mathbb{P})$ of the Functional Kolmogorov Equation (8.6) is also a weak solution.

However, the notion of weak solution is much more general, since equation (8.22) only requires the vertical derivative $\nabla_\omega F(t, X_t)$ to exist in a weak sense, i.e., in $\mathcal{L}^2(X)$, and does not require any continuity of the derivative, only square integrability.

Existence and uniqueness of such weak solutions is much simpler to study than for classical solutions.

Theorem 8.4.5 (Existence and uniqueness of weak solutions)**.** *Let*

$$H \in L^2(C^0([0,T], \mathbb{R}^d), \mathbb{P}).$$

There exists a unique weak solution $F \in \mathbb{W}^{1,2}(\mathbb{P})$ *of the functional Kolmogorov equation* (8.6) *with terminal conditional* $F(T, \cdot) = H(\cdot)$.

Proof. Let M be the martingale component of X and $F\colon \mathcal{W}_T \to \mathbb{R}$ be a regular version of the conditional expectation $E(H|\mathcal{F}_t)$:

$$F(t,\omega) = E\big(H \mid \mathcal{F}_t\big)(\omega), \;\; dt \times d\mathbb{P}\text{-a.e.}$$

Let $Y(t) = F(t, X_t)$. Then $Y \in \mathcal{M}^2(M)$ is a square integrable martingale so, by the martingale representation formula 7.3.4, $Y \in \mathcal{S}^{1,2}(X)$, $F \in \mathbb{W}^{1,2}(\mathbb{P})$, and

$$F(t, X_t) = F(T, X_T) - \int_t^T \nabla_\omega F(u, X_u) dM.$$

Therefore, F satisfies (8.22). This proves existence.

To show uniqueness, we use an 'energy' method. Let F^1, F^2 be weak solutions of (8.6) with terminal condition H. Then, $F = F^1 - F^2 \in \mathbb{W}^{1,2}(\mathbb{P})$ is a weak solution of (8.6) with $F(T, \cdot) = 0$. Let $Y(t) = F(t, X_t)$. Then $Y \in \mathcal{S}^{1,2}(M)$ so, $U(t) = F(T, X_T) - F(t, X_t) - \int_t^T \nabla_M F(u, X_u) dM$ has absolutely continuous paths and

$$\frac{dU}{dt} = -\mathcal{A}F(t, X_t)$$

is well defined. Itô's formula then yields

$$Y(t)^2 = -2\int_t^T Y(u)\nabla_\omega F(u, X_u) dM - 2\int_t^T Y(u)\mathcal{A}F(u, X_u) du + [Y](t) - [Y](T).$$

The first term is a $\mathbb{P}$-local martingale. Let $(\tau_n)_{n\geq 1}$ be an increasing sequence of stopping times such that

$$\int_t^{T\wedge\tau_n} Y(u)\cdot\nabla_\omega F(u,X_u)dM$$

is a martingale. Then, for any $n\geq 1$,

$$E\biggl(\int_t^{T\wedge\tau_n} Y(u)\nabla_\omega F(u,X_u)dM\biggr)=0.$$

Therefore, for any $n\geq 1$,

$$E\bigl(Y(t\wedge\tau_n)^2\bigr)=2E\biggl(\int_t^{T\wedge\tau_n} Y(u)\mathcal{A}F(u,X_u)du\biggr)+E\bigl([Y](t\wedge\tau_n)-[Y](T\wedge\tau_n)\bigr).$$

Given that F is a weak solution of (8.6),

$$E\biggl(\int_t^{T\wedge\tau_n} Y(u)\mathcal{A}F(u,X_u)du\biggr)=E\biggl(\int_t^{T} Y(u)1_{[0,\tau_n]}\mathcal{A}F(u,X_u)du\biggr)=0,$$

because $Y(u)1_{[0,\tau_n]}\in\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})$. Thus,

$$E\bigl(Y(t\wedge\tau_n)^2\bigr)=E\bigl([Y](t\wedge\tau_n)-[Y](T\wedge\tau_n)\bigr),$$

which is a positive increasing function of t; so,

$$\forall n\geq 1,\quad E\bigl(Y(t\wedge\tau_n)^2\bigr)\leq E\bigl(Y(T\wedge\tau_n)^2\bigr).$$

Since $Y\in\mathcal{L}^2(\mathbb{F},dt\times d\mathbb{P})$ we can use a dominated convergence argument to take $n\to\infty$ and conclude that $0\leq E\left(Y(t)^2\right)\leq E\left(Y(T)^2\right)=0$. Therefore, $Y=0$, i.e., $F^1(t,\omega)=F^2(t,\omega)$ outside a $\mathbb{P}$-evanescent set. □

Note that uniqueness holds outside a $\mathbb{P}$-evanescent set: there is no assertion on uniqueness outside the support of $\mathbb{P}$.

The flexibility of the notion of weak solution comes from the following characterization of square integrable martingale functionals as weak solutions, which is an analog of Theorem 8.1.6 but *without* any differentiability requirement.

Proposition 8.4.6 (Characterization of harmonic functionals)**.** *Let $F\in\mathcal{L}^2(dt\times d\mathbb{P})$ be a square integrable non-anticipative functional. Then, $Y(t)=F(t,X_t)$ is a $\mathbb{P}$-martingale* if and only if *F is a weak solution of the Functional Kolmogorov Equation* (8.6).

Proof. Let F be a weak solution to (8.6). Then, $S=F(X)\in\mathcal{S}^{1,2}(X)$ so S is weakly differentiable with $\nabla_M S\in\mathcal{L}^2(X)$, $N=\int_0^{\cdot}\nabla_M S\cdot dM\in\mathcal{M}^2(X)$ is a square integrable martingale, and $A = S - N \in \mathcal{A}^2(\mathbb{F})$. Since F verifies (8.22), we have

$$\forall \phi \in \mathcal{L}^2(dt \times d\mathbb{P}), \quad E\left(\int_0^T \phi(t) A(t) dt \right) = 0,$$

so $A(t) = 0$ $dt \times d\mathbb{P}$-a.e.; therefore, S is a martingale.

Conversely, let $F \in \mathcal{L}^2(dt \times d\mathbb{P})$ be a non-anticipative functional such that $S(t) = F(t, X_t)$ is a martingale. Then $S \in \mathcal{M}^2(X)$ so, by Theorem 7.3.4, S is weakly differentiable and $U(t) = F(T, X_T) - F(t, X_t) - \int_t^T \nabla_M S \cdot dM = 0$. Hence, F verifies (8.22) so, F is a weak solution of the functional Kolmogorov equation with terminal condition $F(T, \cdot)$. □

This result characterizes *all* square integrable *harmonic functionals* as weak solutions of the Functional Kolmogorov Equation (8.6) and thus illustrates the relevance of Definition 8.4.4.

Comments and references

Path dependent Kolmogorov equations first appeared in the work of Dupire [21]. Uniqueness and comparison of classical solutions were first obtained in [11, 32]. The path dependent HJB equation and its relation with non-Markovian stochastic control problems was first explored in [32]. The relation between FBSDEs and path dependent PDEs is an active research topic and the fully nonlinear case, corresponding to non-Markovian optimal control and optimal stopping problems, is currently under development, using an extension of the notion of *viscosity solution* to the functional case, introduced by Ekren et al. [22, 23]. A full discussion of this notion is beyond the scope of these lecture notes and we refer to [10] and recent work by Ekren et al. [22, 23], Peng [60, 61] and Cosso [14].

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